



Renormalized solutions for degenerate elliptic–parabolic problems with nonlinear dynamical boundary conditions and L^1 -data

F. Andreu^a, N. Igbida^b, J.M. Mazón^{a,*}, J. Toledo^a

^a *Departamento de Análisis Matemático, Universitat de València, Dr. Moliner 50, 46100 Burjassot (Valencia), Spain*

^b *LAMFA, CNRS-UMR 6140, Université de Picardie Jules Verne, 33 rue Saint Leu, 80038 Amiens, France*

Received 6 July 2006; revised 7 November 2007

Abstract

We consider a degenerate elliptic–parabolic problem with nonlinear dynamical boundary conditions. Assuming L^1 -data, we prove existence and uniqueness in the framework of renormalized solutions. Particular instances of this problem appear in various phenomena with changes of phase like multiphase Stefan problems and in the weak formulation of the mathematical model of the so-called Hele–Shaw problem. Also, the problem with non-homogeneous Neumann boundary condition is included.

© 2008 Elsevier Inc. All rights reserved.

Keywords: Degenerate elliptic–parabolic problems; Renormalized solutions; Multiphase Stefan problems; Hele–Shaw problem

1. Introduction

In this paper we obtain existence and uniqueness of renormalized solutions for a degenerate elliptic–parabolic problem with nonlinear dynamical boundary condition of the form

$$P_{\gamma,\beta}(f, g, z_0, w_0) \quad \begin{cases} z_t - \operatorname{div} \mathbf{a}(x, Du) = f, & z \in \gamma(u) \text{ in } Q_T :=]0, T[\times \Omega, \\ w_t + \mathbf{a}(x, Du) \cdot \eta = g, & w \in \beta(u) \text{ on } S_T :=]0, T[\times \partial\Omega, \\ z(0) = z_0 & \text{ in } \Omega, \quad w(0) = w_0 & \text{ in } \partial\Omega, \end{cases}$$

* Corresponding author.

E-mail addresses: fuensanta.andreu@uv.es (F. Andreu), noureddine.igbida@u-picardie.fr (N. Igbida), mazon@uv.es (J.M. Mazón), toledo@uv.es (J. Toledo).

where $T > 0$, Ω is a bounded domain in \mathbb{R}^N with smooth boundary $\partial\Omega$, $v_0 \in L^1(\Omega)$, $w_0 \in L^1(\partial\Omega)$, $f \in L^1(0, T; L^1(\Omega))$, $g \in L^1(0, T; L^1(\partial\Omega))$ and η is the unit outward normal on $\partial\Omega$. Here the function $\mathbf{a}: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function satisfying the classical Leray–Lions conditions. The nonlinearities γ and β are maximal monotone graphs in \mathbb{R}^2 (see [20]) such that $0 \in \gamma(0)$, $\text{Dom}(\gamma) = \mathbb{R}$, and $0 \in \beta(0)$. In particular, γ and β may be multivalued and this allows to include the Dirichlet boundary condition (taking β to be the monotone graph $\{0\} \times \mathbb{R}$), the *non-homogeneous* Neumann boundary condition (taking β to be the monotone graph $\beta(r) = 0$ for all $r \in \mathbb{R}$), as well as many other nonlinear fluxes on the boundary that occur in some problems in mechanics and physics (see [28] or [19]). Note also that, since γ may be multivalued, problems of type $P_{\gamma, \beta}(f, g, z_0, w_0)$ appear in various phenomena with changes of phase like multiphase Stefan problem (see [25]) and in the weak formulation of the mathematical model of the so-called Hele–Shaw problem (see [26] and [29]), for which γ is the Heaviside maximal monotone graph. Also, if $\gamma(r) = 0$ for all $r \in \mathbb{R}$, we consider an elliptic problem with nonlinear dynamical boundary condition.

The dynamical boundary conditions, although not too widely considered in the mathematical literature, are very natural in many mathematical models as heat transfer in a solid in contact with moving fluid, thermoelasticity, diffusion phenomena, problems in fluid dynamics, etc. (see [11,23,30,43] and the references therein). These dynamical boundary conditions also appear in the study of the Stefan problem when the boundary material has a large thermal conductivity and sufficiently small thickness. Hence, the boundary material is regarded as the boundary of the domain. For instance, this is the case if one considers an iron ball in which water and ice coexist. For more details about these physical considerations one can see for instance [1]. They also appear in the study of the Hele–Shaw problem. Recall that, in [26] the authors give the weak formulation of the problem in the form of a nonlinear degenerate parabolic problem, governed by the Laplace operator and the multivalued Heaviside function, with static boundary condition. From the physical point of view they assume that the prescribed value of the flux on the boundary is known, but in some practical situations, it may be not possible to prescribe or to control the exact value of the flux on the boundary. In [42] (see also [43]), the authors consider the case of nonlocal dynamical boundary conditions and use variational methods to solve the problem. In the present paper, we cover the case of general nonlinear diffusion and local dynamical boundary conditions. Notice, that general nonlinear diffusion operators of Leray–Lions type, different from the Laplacian, appear when one deals with non-Newtonian fluids (see, e.g., [9,37,38] and the references therein for the case of Hele–Shaw problem with non-Newtonian fluids). Another interesting application we have in mind concerns the filtration equation with dynamical boundary conditions (see, e.g., [44]), which appears for example in the study of rainfall infiltration through the soil, when the accumulation of the water on the ground surfaces caused by the saturation of the surface layer is taken into account. Observe that β may be such that $\text{Ran}(\beta)$ is different from \mathbb{R} , so that we cover the case where the boundary conditions are either dynamical or static with respect to the values of w in the problem under consideration. This is the situation where the saturation happens only for values of w in a subinterval of \mathbb{R} .

There is an extensive literature for doubly nonlinear problems with homogeneous Dirichlet boundary conditions (see [2,3,10,15,17,21,34] and the references therein). Nevertheless, to our knowledge, there is little literature on problem $P_{\gamma, \beta}(f, g, z_0, w_0)$ as we pointed out in [5], where existence and uniqueness of weak solutions of this problem have been obtained for $L^{p'}$ -data. Our aim in this paper is to prove existence and uniqueness of solutions for L^1 -data of $P_{\gamma, \beta}(f, g, z_0, w_0)$. There are mainly two types of difficulties in studying this kind of problems, the nonlinearities γ and β and the consideration of L^1 -data so that finite energy solutions could

not be expected. To solve this last difficulty, the framework of renormalized solutions, which was originally introduced in [27] for transport equations, has proved to be a powerful approach to study large class of second order PDEs (see [3,7,16,17,22] and the references therein).

Another main difficulty when dealing with doubly nonlinear parabolic problems is the uniqueness. For the Laplace operator, thanks to the linearity of the operator, the problem can be solved by using suitable test functions with respect to u (see for instance [33]). For nonlinear operators this kind of argument turns out to be nonuseful. In [15], for an elliptic–parabolic problem with Dirichlet boundary condition, it is shown that the notion of integral solution [12] is a very useful tool to prove uniqueness (see also [32] for non-homogeneous and time dependent Neumann boundary conditions). For general nonlinearities, even for homogeneous Dirichlet boundary condition, the question of uniqueness is more difficult and most of the arguments used in the literature are based on doubling variables methods (see for instance [8,17,21,22,34] and the references therein). In [5] we have shown that the notion of integral solution is a very useful tool to prove uniqueness of weak solutions of problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ for $L^{p'}$ -data. In this paper, we use the same method to prove uniqueness of renormalized solutions of problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ for L^1 -data.

We also want to point out that our existence and uniqueness proofs work without any continuity assumptions on γ^{-1} or β^{-1} and any hypothesis about the jumps of γ or β . For the existence of the renormalized solution, we use a monotone approximation of f , g , z_0 and w_0 , by L^∞ functions $f_{m,n}$, $g_{m,n}$, $z_{0,m,n}$ and $w_{0,m,n}$. So that, by using the results of [5], the problem has a unique weak solution $(z_{m,n}, w_{m,n})$. Thanks to the nonlinear semigroup theory (see [14,45]), the results of [4] concerning the stationary problem associated with $P_{\gamma,\beta}(f, g, z_0, w_0)$, it is not difficult to get the L^1 convergence of $(z_{m,n}, w_{m,n})$. Nevertheless, the characterization of the limit of $(z_{m,n}, w_{m,n})$ in terms of the partial differential equation is very technical due to the fact that the problem is doubly nonlinear. For the convergence of $u_{m,n}$ (see the proof of Theorem 2.6), we use the monotonicity with respect to m and n , as it was used in [3], and for the identification of the limit equation we use Landes approximation (see [39]). Recall that this kind of arguments was also used in [3] for elliptic–parabolic problems and in [35] for degenerate parabolic problems of Stefan type. Here we extend these arguments to our general setting (other kind of arguments may be found in [17]). For the uniqueness, we show that renormalized solutions are *integral solutions*, concept due to Ph. Bénilan (see [12,14]). In other words, we show that renormalized solutions satisfy a contraction property with respect to stationary solutions. The main difficulties here are due to the nonlinear and non-homogeneous boundary conditions and to the jumps of γ and β . In [17], to obtain a contraction principle for a similar problem in the case of Dirichlet boundary condition ($\beta = \{0\} \times \mathbb{R}$), and for γ having a set of jumps without density points, the authors give an improvement of the “hole filling” argument of [21], using the doubling variable technique in time and a very useful choice of test functions. This technique can be adapted to our problem. Now, as in [5], by the nonlinear semigroup theory, we are able to simplify the proof of uniqueness without using the doubling variable technique in time and without imposing any condition on the jumps of γ and β .

Let us briefly summarize the contents of the paper. In Section 2 we fix the notation and give some preliminaries; we also give the concept of renormalized solution for the problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ and state the existence and uniqueness result for renormalized solutions of problem $P_{\gamma,\beta}(f, g, z_0, w_0)$. In Section 3 we show the existence of renormalized solutions and finally in Section 4 we prove the uniqueness of renormalized solutions.

2. Preliminaries and main result

In this section, after some preliminaries, we introduce the concept of renormalized solution for problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ and we state the existence and uniqueness result for this type of solutions.

Throughout the paper, $\Omega \subset \mathbb{R}$ is a bounded domain with smooth boundary $\partial\Omega$, $p > 1$, γ and β are maximal monotone graphs in \mathbb{R}^2 such that $\text{Dom}(\gamma) = \mathbb{R}$, $0 \in \gamma(0) \cap \beta(0)$ and the Carathéodory function $\mathbf{a}: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ satisfies

- (H₁) there exists $\lambda > 0$ such that $\mathbf{a}(x, \xi) \cdot \xi \geq \lambda |\xi|^p$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$,
- (H₂) there exist $c > 0$ and $\varrho \in L^{p'}(\Omega)$ such that $|\mathbf{a}(x, \xi)| \leq c(\varrho(x) + |\xi|^{p-1})$ for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$, where $p' = \frac{p}{p-1}$,
- (H₃) $(\mathbf{a}(x, \xi) - \mathbf{a}(x, \eta)) \cdot (\xi - \eta) > 0$ for a.e. $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^N$, $\xi \neq \eta$.

The hypotheses (H₁)–(H₃) are classical in the study of nonlinear operators in divergence form (see [41] or [13]). The model example of function \mathbf{a} satisfying these hypotheses is $\mathbf{a}(x, \xi) = |\xi|^{p-2}\xi$. The corresponding operator is the p -Laplacian operator $\Delta_p(u) = \text{div}(|Du|^{p-2}Du)$.

In [13], the authors introduce the set

$$\mathcal{T}^{1,p}(\Omega) = \{u: \Omega \rightarrow \mathbb{R} \text{ measurable such that } T_k(u) \in W^{1,p}(\Omega) \forall k > 0\},$$

where $T_k(s) = \sup(-k, \inf(s, k))$. They also prove that given $u \in \mathcal{T}^{1,p}(\Omega)$, there exists a unique measurable function $v: \Omega \rightarrow \mathbb{R}^N$ such that

$$DT_k(u) = v\chi_{\{|u| < k\}} \quad \forall k > 0.$$

This function v will be denoted by Du . It is clear that if $u \in W^{1,p}(\Omega)$, then $v \in L^p(\Omega)$ and $v = Du$ in the usual sense.

We denote $\mathcal{T}_\tau^{1,p}(\Omega)$ the set of functions u in $\mathcal{T}^{1,p}(\Omega)$ such that there exists a measurable function w on $\partial\Omega$ with $T_k(w) = \text{tr}(T_k(u))$ a.e. on $\partial\Omega$ for all $k > 0$, where tr is the usual $W^{1,p}$ -trace. The function w is the trace of u in a generalized sense. In the sequel, the trace of $u \in \mathcal{T}_\tau^{1,p}(\Omega)$ on $\partial\Omega$ will be denoted by u .

For a maximal monotone graph ϑ in $\mathbb{R} \times \mathbb{R}$, its *main section* ϑ^0 is defined by

$$\vartheta^0(s) := \begin{cases} \text{the element of minimal absolute value of } \vartheta(s) & \text{if } \vartheta(s) \neq \emptyset, \\ +\infty & \text{if } [s, +\infty) \cap \text{Dom}(\vartheta) = \emptyset, \\ -\infty & \text{if } (-\infty, s] \cap \text{Dom}(\vartheta) = \emptyset. \end{cases}$$

We shall denote $\vartheta_- := \inf \text{Ran}(\vartheta)$ and $\vartheta_+ := \sup \text{Ran}(\vartheta)$. If $0 \in \text{Dom}(\vartheta)$, $j_\vartheta(r) = \int_0^r \vartheta^0(s) ds$ defines a convex l.s.c. function such that $\vartheta = \partial j_\vartheta$. If j_ϑ^* is the Legendre transformation of j_ϑ then $\vartheta^{-1} = \partial j_\vartheta^*$.

For the maximal monotone graphs γ and β , we shall denote

$$\mathcal{R}_{\gamma,\beta}^+ := \gamma_+|\Omega| + \beta_+|\partial\Omega|, \quad \mathcal{R}_{\gamma,\beta}^- := \gamma_-|\Omega| + \beta_-|\partial\Omega|.$$

In the sequel, we suppose $\mathcal{R}_{\gamma,\beta}^- < \mathcal{R}_{\gamma,\beta}^+$ and we write $\mathcal{R}_{\gamma,\beta} :=]\mathcal{R}_{\gamma,\beta}^-, \mathcal{R}_{\gamma,\beta}^+[$.

It is said that \mathbf{a} is *smooth* (see [6] and [4]) if, for any $\phi \in L^\infty(\Omega)$ such that there exists a bounded weak solution u of the homogeneous Dirichlet problem

$$(D) \quad \begin{cases} -\operatorname{div} \mathbf{a}(x, Du) = \phi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

there exists $g \in L^1(\partial\Omega)$ such that u is also a weak solution of the Neumann problem

$$(N) \quad \begin{cases} -\operatorname{div} \mathbf{a}(x, Du) = \phi & \text{in } \Omega, \\ \mathbf{a}(x, Du) \cdot \eta = g & \text{on } \partial\Omega. \end{cases}$$

Functions \mathbf{a} corresponding to linear operators with smooth coefficients and p -Laplacian type operators are smooth (see [19] and [40]).

The following integration by parts formula, which is a slight modification of [5, Lemma 4.1], will play an important role in our arguments. We denote by (\cdot, \cdot) the pairing between $(W^{1,p}(\Omega))'$ and $W^{1,p}(\Omega)$.

Lemma 2.1. (See [5].) Let ϑ and ϱ be maximal monotone graphs in \mathbb{R}^2 . Let $z \in C([0, T] : L^1(\Omega))$, $w \in C([0, T] : L^1(\partial\Omega))$, $F \in L^{p'}(0, T; (W^{1,p}(\Omega))')$, $f \in L^1(0, T; L^1(\Omega))$ and $g \in L^1(0, T; L^1(\partial\Omega))$ such that

$$\int_0^T \int_\Omega z \psi_t dx dt + \int_0^T \int_{\partial\Omega} w \psi_t d\sigma dt = \int_0^T (F(t), \psi(t)) dt + \int_0^T \int_\Omega f \psi dx dt + \int_0^T \int_{\partial\Omega} g \psi d\sigma dt$$

for any $\psi \in W^{1,1}(0, T; W^{1,1}(\Omega) \cap L^\infty(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega)) \cap L^\infty(0, T; L^\infty(\Omega))$, $\psi(0) = \psi(T) = 0$. Then,

$$\begin{aligned} & \int_0^T \int_\Omega \left(\int_0^{z(t)} H(x, (\vartheta^{-1})^0(s)) ds \right) \psi_t dx dt + \int_0^T \int_{\partial\Omega} \left(\int_0^{w(t)} H(x, (\varrho^{-1})^0(s)) ds \right) \psi_t d\sigma dt \\ &= \int_0^T (F(t), H(x, u(t)) \psi(t)) dt + \int_0^T \int_\Omega f H(x, u) \psi dx dt + \int_0^T \int_{\partial\Omega} g H(x, u) \psi d\sigma dt, \end{aligned}$$

being $u \in L^p(0, T; W^{1,p}(\Omega))$ such that $z \in \vartheta(u)$ a.e. in Q_T and $w \in \varrho(u)$ a.e. in S_T , $H(x, r)$ a bounded Carathéodory function of bounded variation in r , such that $H(\cdot, u(\cdot, \cdot)) \in L^p(0, T; W^{1,p}(\Omega))$, and $\psi \in \mathcal{D}([0, T] \times \mathbb{R}^N)$.

We now recall the concept of weak solution for problem $P_{\gamma,\beta}(f, g, z_0, w_0)$ and state the existence and uniqueness result given in [5] for such solutions.

Definition 2.2. Given $f \in L^1(0, T; L^1(\Omega))$, $g \in L^1(0, T; L^1(\partial\Omega))$, $z_0 \in L^1(\Omega)$ and $w_0 \in L^1(\partial\Omega)$, a *weak solution* of $P_{\gamma,\beta}(f, g, z_0, w_0)$ in $[0, T]$ is a couple (z, w) such that $z \in C([0, T] : L^1(\Omega))$, $w \in C([0, T] : L^1(\partial\Omega))$, $z(0) = z_0$, $w(0) = w_0$ and there exists $u \in L^p(0, T; W^{1,p}(\Omega))$ such that $z \in \gamma(u)$ a.e. in Q_T , $w \in \beta(u)$ a.e. on S_T and

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} z(t) \xi \, dx + \frac{d}{dt} \int_{\partial\Omega} w(t) \xi \, d\sigma + \int_{\Omega} \mathbf{a}(x, Du(t)) \cdot D\xi \, dx \\ &= \int_{\Omega} f(t) \xi \, dx + \int_{\partial\Omega} g(t) \xi \, d\sigma \end{aligned} \quad (1)$$

in $\mathcal{D}'([0, T])$ for any $\xi \in C^1(\bar{\Omega})$.

Theorem 2.3. (See [5].) Assume $\text{Dom}(\gamma) = \mathbb{R}$ and assume either $\text{Dom}(\beta) = \mathbb{R}$ or \mathbf{a} smooth. Let $T > 0$. Let $f \in L^{p'}(0, T; L^{p'}(\Omega))$, $g \in L^{p'}(0, T; L^{p'}(\partial\Omega))$, $z_0 \in L^{p'}(\Omega)$ and $w_0 \in L^{p'}(\partial\Omega)$ such that

$$\gamma_- \leq z_0 \leq \gamma_+, \quad \beta_- \leq w_0 \leq \beta_+, \quad (2)$$

$$\int_{\Omega} j_{\gamma}^*(z_0) \, dx + \int_{\partial\Omega} j_{\beta}^*(w_0) \, d\sigma < +\infty \quad (3)$$

and

$$\int_{\Omega} z_0 \, dx + \int_{\partial\Omega} w_0 \, d\sigma + \int_0^t \left(\int_{\Omega} f(s) \, dx + \int_{\partial\Omega} g(s) \, d\sigma \right) ds \in \mathcal{R}_{\gamma, \beta} \quad (4)$$

for all $t \in [0, T]$. Then, there exists a unique weak solution (z, w) of problem $P_{\gamma, \beta}(f, g, z_0, w_0)$ in $[0, T]$.

Moreover, the following L^1 -contraction principle holds. For $i = 1, 2$, let $f_i \in L^1(0, T; L^1(\Omega))$, $g_i \in L^1(0, T; L^1(\partial\Omega))$, $z_{i0} \in L^1(\Omega)$ and $w_{i0} \in L^1(\partial\Omega)$; let (z_i, w_i) be a weak solution in $[0, T]$ of problem $P_{\gamma, \beta}(f_i, g_i, z_{i0}, w_{i0})$, $i = 1, 2$. Then

$$\begin{aligned} & \int_{\Omega} (z_1(t) - z_2(t))^+ \, dx + \int_{\partial\Omega} (w_1(t) - w_2(t))^+ \, d\sigma \\ & \leq \int_{\Omega} (z_{10} - z_{20})^+ \, dx + \int_{\partial\Omega} (w_{10} - w_{20})^+ \, d\sigma \\ & \quad + \int_0^t \int_{\Omega} (f_1(s) - f_2(s))^+ \, dx \, ds + \int_0^t \int_{\partial\Omega} (g_1(s) - g_2(s))^+ \, d\sigma \, ds \end{aligned}$$

for every $t \in [0, T]$.

Let us give the concept of renormalized solution.

Definition 2.4. Given $f \in L^1(0, T; L^1(\Omega))$, $g \in L^1(0, T; L^1(\partial\Omega))$, $z_0 \in L^1(\Omega)$ and $w_0 \in L^1(\partial\Omega)$, a renormalized solution of $P_{\gamma, \beta}(f, g, z_0, w_0)$ in $[0, T]$ is a couple (z, w) , $z \in C([0, T]; L^1(\Omega))$, $w \in C([0, T]; L^1(\partial\Omega))$, $z(0) = z_0$, $w(0) = w_0$, for which there exists a measurable

function u in $]0, T[\times \Omega$, $u(t) \in \mathcal{T}_\tau^{1,p}(\Omega)$ a.e. $t \in]0, T[$, such that $T_k(u) \in L^p(0, T; W^{1,p}(\Omega))$ for all $k > 0$, $z \in \gamma(u)$ a.e. in Q_T , $w \in \beta(u)$ a.e. on S_T ,

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left(\int_0^{z(t)} H((\gamma^{-1})^0(s)) ds \right) \xi dx + \frac{d}{dt} \int_{\partial\Omega} \left(\int_0^{w(t)} H((\beta^{-1})^0(s)) ds \right) \xi d\sigma \\ & + \int_{\Omega} \mathbf{a}(x, Du(t)) \cdot D(H(u(t))\xi) dx \\ & = \int_{\Omega} f(t)H(u(t))\xi dx + \int_{\partial\Omega} g(t)H(u(t))\xi d\sigma \end{aligned} \quad (5)$$

in $\mathcal{D}'(]0, T[)$, for any $\xi \in C^1(\bar{\Omega})$ and any Lipschitz continuous function $H: \mathbb{R} \rightarrow \mathbb{R}$ of compact support, and

$$\lim_{n \rightarrow +\infty} \int_{\{(t,x) \in Q_T: n \leq |u(t,x)| \leq n+1\}} \mathbf{a}(x, Du) \cdot Du dx dt = 0. \quad (6)$$

Remark 2.5. (i) In (5) and (6) every term is well defined. Observe that the third term of the left-hand side of (5) has to be understood as

$$\int_{\Omega} \mathbf{a}(x, Du) \cdot D(H(u)\xi) dx = \int_{\Omega} \mathbf{a}(x, DT_M(u)) \cdot D(H(T_M(u))\xi) dx,$$

where $M > 0$ is such that $\text{supp}(H) \subset [-M, M]$. Similarly, the integral in (6) has to be understood as

$$\int_{\{(t,x) \in Q_T: n \leq |u(t,x)| \leq n+1\}} \mathbf{a}(x, DT_{n+1}(u)) \cdot DT_{n+1}(u) dx dt.$$

(ii) A renormalized solution satisfies

$$\int_{\Omega} z(t) dx + \int_{\partial\Omega} w(t) d\sigma = \int_{\Omega} z_0 dx + \int_{\partial\Omega} w_0 d\sigma + \int_0^t \left(\int_{\Omega} f(s) dx + \int_{\partial\Omega} g(s) d\sigma \right) ds \quad (7)$$

for all $t \in [0, T]$.

(iii) A weak solution in the sense of Definition 2.2 is a renormalized solution. In fact, if (z, w) is a weak solution of $P_{\gamma, \beta}(f, g, z_0, w_0)$ in $[0, T]$, then there exists $u \in L^p(0, T; W^{1,p}(\Omega))$ such that $z \in \gamma(u)$ a.e. in Q_T , $w \in \beta(u)$ a.e. on S_T and

$$\begin{aligned}
\int_0^T \int_{\Omega} \mathbf{a}(x, Du) \cdot D\psi \, dx \, dt &= \int_0^T \int_{\Omega} z \psi_t \, dx \, dt + \int_0^T \int_{\partial\Omega} w \psi_t \, d\sigma \, dt \\
&\quad + \int_0^T \int_{\Omega} f \psi \, dx \, dt + \int_0^T \int_{\partial\Omega} g \psi \, d\sigma \, dt
\end{aligned} \tag{8}$$

for any $\psi \in W^{1,1}(0, T; W^{1,1}(\Omega) \cap L^\infty(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$, $\psi(0) = \psi(T) = 0$. Then, by Lemma 2.1, we have

$$\begin{aligned}
&\int_0^T \int_{\Omega} \int_0^{z(t)} H((\gamma^{-1})^0(s)) \, ds \, \psi_t \, dx \, dt + \int_0^T \int_{\partial\Omega} \int_0^{w(t)} H((\beta^{-1})^0(s)) \, ds \, \psi_t \, d\sigma \, dt \\
&= \int_0^T \int_{\Omega} \mathbf{a}(x, Du) \cdot D(H(u)\psi) \, dx \, dt \\
&\quad - \int_0^T \int_{\Omega} f H(u) \psi \, dx \, dt - \int_0^T \int_{\partial\Omega} g H(u) \psi \, d\sigma \, dt
\end{aligned} \tag{9}$$

for any $H: \mathbb{R} \rightarrow \mathbb{R}$ Lipschitz continuous of compact support and $\psi(t, x) = \varphi(t)\xi(x)$, with $\varphi \in \mathcal{D}([0, T])$ and $\xi \in C^1(\bar{\Omega})$. Hence (5) holds. Moreover, since $u \in L^p(0, T; W^{1,p}(\Omega))$, (6) also holds, and consequently (z, w) is a renormalized solution of $P_{\gamma,\beta}(f, g, z_0, w_0)$ in $[0, T]$.

(iv) If u is a renormalized solution such that $u \in L^p(0, T; W^{1,p}(\Omega))$, u is a weak solution in the sense of Definition 2.2.

The main result of this paper is the following existence and uniqueness theorem.

Theorem 2.6. Assume $\text{Dom}(\gamma) = \mathbb{R}$ and assume either $\text{Dom}(\beta) = \mathbb{R}$ or \mathbf{a} smooth. Let $T > 0$.

(i) Let $f \in L^1(0, T; L^1(\Omega))$, $g \in L^1(0, T; L^1(\partial\Omega))$, $z_0 \in L^1(\Omega)$ and $w_0 \in L^1(\partial\Omega)$ such that

$$\gamma_- \leq z_0 \leq \gamma_+, \quad \beta_- \leq w_0 \leq \beta_+ \tag{10}$$

and

$$\int_{\Omega} z_0 \, dx + \int_{\partial\Omega} w_0 \, d\sigma + \int_0^t \left(\int_{\Omega} f(s) \, dx + \int_{\partial\Omega} g(s) \, d\sigma \right) ds \in \mathcal{R}_{\gamma,\beta} \tag{11}$$

for all $t \in [0, T]$. Then, there exists a unique renormalized solution (z, w) of $P_{\gamma,\beta}(f, g, z_0, w_0)$ in $[0, T]$.

(ii) Moreover, the following L^1 -contraction principle holds. For $i = 1, 2$, let $f_i \in L^1(0, T; L^1(\Omega))$, $g_i \in L^1(0, T; L^1(\partial\Omega))$, $z_{i0} \in L^1(\Omega)$ and $w_{i0} \in L^1(\partial\Omega)$; and let (z_i, w_i) be a renormalized solution in $[0, T]$ of $P_{\gamma,\beta}(f_i, g_i, z_{i0}, w_{i0})$, $i = 1, 2$. Then

$$\begin{aligned}
& \int_{\Omega} (z_1(t) - z_2(t))^+ dx + \int_{\partial\Omega} (w_1(t) - w_2(t))^+ d\sigma \\
& \leq \int_{\Omega} (z_{10} - z_{20})^+ dx + \int_{\partial\Omega} (w_{10} - w_{20})^+ d\sigma \\
& \quad + \int_0^t \int_{\Omega} (f_1(s) - f_2(s))^+ dx ds + \int_0^t \int_{\partial\Omega} (g_1(s) - g_2(s))^+ d\sigma ds
\end{aligned} \tag{12}$$

for every $t \in [0, T]$.

To prove the above theorem we use the nonlinear semigroup theory (see [12,14] or [24]).

Remark 2.7. We recall that in the case $\beta = 0$, for the Laplacian operator and γ the multivalued Heaviside function (i.e., for the Hele–Shaw problem), existence and uniqueness of weak solutions for this problem is known to be true only if

$$\int_{\Omega} z_0 dx + \int_0^t \left(\int_{\Omega} f(s) dx + \int_{\partial\Omega} g(s) d\sigma \right) ds \in (0, |\Omega|) \quad \text{for any } t \in [0, T]$$

(see [36, Theorem 3.1 and Example 8.1], see also [31]). The same example works for renormalized solutions, so condition (11) is necessary.

3. Existence of renormalized solutions

In this section we prove the existence part of Theorem 2.6. We use the following lemma proved in [5, Lemma 4.2].

Lemma 3.1. (See [5].) Let $\{u_n\}_{n \in \mathbb{N}} \subset W^{1,p}(\Omega)$, $\{z_n\}_{n \in \mathbb{N}} \subset L^1(\Omega)$, $\{w_n\}_{n \in \mathbb{N}} \subset L^1(\partial\Omega)$ such that, for every $n \in \mathbb{N}$, $z_n \in \gamma(u_n)$ a.e. in Ω and $w_n \in \beta(u_n)$ a.e. in $\partial\Omega$. Let us suppose that

(i) if $\mathcal{R}_{\gamma,\beta}^+ = +\infty$, there exists $M > 0$ such that

$$\int_{\Omega} z_n^+ dx + \int_{\partial\Omega} w_n^+ d\sigma < M \quad \forall n \in \mathbb{N};$$

(ii) if $\mathcal{R}_{\gamma,\beta}^+ < +\infty$, there exists $M \in \mathbb{R}$ such that

$$\int_{\Omega} z_n dx + \int_{\partial\Omega} w_n d\sigma < M < \mathcal{R}_{\gamma,\beta}^+$$

and

$$\lim_{L \rightarrow +\infty} \left(\int_{\{x \in \Omega: z_n(x) < -L\}} |z_n| dx + \int_{\{x \in \partial\Omega: w_n(x) < -L\}} |w_n| d\sigma \right) = 0$$

uniformly in $n \in \mathbb{N}$.

Then, there exists a constant $C = C(M)$ such that

$$\|u_n^+\|_{L^p(\Omega)} \leq C(\|Du_n^+\|_{L^p(\Omega)} + 1) \quad \forall n \in \mathbb{N}.$$

Proof of Theorem 2.6. (Existence) We divide the proof in several steps.

Step 1: Approximate problems. For $f \in L^1(0, T; L^1(\Omega))$, $g \in L^1(0, T; L^1(\partial\Omega))$, $z_0 \in L^1(\Omega)$ and $w_0 \in L^1(\partial\Omega)$ satisfying (10) and (11), let

$$\begin{aligned} f_{m,n} &= \sup\{\inf\{m, f\}, -n\}, & g_{m,n} &= \sup\{\inf\{m, g\}, -n\}, \\ z_{0m,n} &= \sup\{\inf\{m, z_0\}, -n\} \quad \text{and} \quad w_{0m,n} = \sup\{\inf\{m, w_0\}, -n\}, \end{aligned}$$

where $m, n \in \mathbb{N}$, and consider the approximate problems

$$P_{\gamma,\beta}(f_{m,n}, g_{m,n}, z_{0m,n}, w_{0m,n}).$$

It is clear that for m, n large enough, $f_{m,n}, g_{m,n}, z_{0m,n}, w_{0m,n}$ satisfy (2)–(4), in fact, there exist $r_1, r_2 \in \mathbb{R}$ such that, for any m, n large enough and any $t \in [0, T]$,

$$\begin{aligned} \mathcal{R}_{\gamma,\beta}^- &< r_1 \leq \int_{\Omega} z_{0m,n} dx + \int_{\partial\Omega} w_{0m,n} d\sigma \\ &+ \int_0^t \left(\int_{\Omega} f_{m,n}(s) dx + \int_{\partial\Omega} g_{m,n}(s) d\sigma \right) ds \leq r_2 < \mathcal{R}_{\gamma,\beta}^+. \end{aligned} \quad (13)$$

Therefore, by Theorem 2.3, there exists a unique weak solution $(z_{m,n}, w_{m,n})$ of problem $P_{\gamma,\beta}(f_{m,n}, g_{m,n}, z_{0m,n}, w_{0m,n})$, so there exists $u_{m,n} \in L^p(0, T; W^{1,p}(\Omega))$ such that $z_{m,n} \in \gamma(u_{m,n})$ a.e. in $\Omega \times]0, T[$, $w_{m,n} \in \beta(u_{m,n})$ a.e. in $\partial\Omega \times]0, T[$, and

$$\begin{aligned} \int_0^T \int_{\Omega} \mathbf{a}(x, Du_{m,n}) \cdot D\psi dx dt &= \int_0^T \int_{\Omega} z_{m,n} \psi_t dx dt + \int_0^T \int_{\partial\Omega} w_{m,n} \psi_t d\sigma dt \\ &+ \int_0^T \int_{\Omega} f_{m,n} \psi dx dt + \int_0^T \int_{\partial\Omega} g_{m,n} \psi d\sigma dt \end{aligned} \quad (14)$$

for any $\psi \in W^{1,1}(0, T; W^{1,1}(\Omega) \cap L^\infty(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$, $\psi(0) = \psi(T) = 0$. Since $f_{m,n}, g_{m,n}, z_{0m,n}$ and $w_{0m,n}$ are monotone nondecreasing in m and monotone nonincreasing in n , by results of [4] and [5], we can also consider that so are $u_{m,n}, z_{m,n}$ and $w_{m,n}$. Therefore, there exists a subsequence $\{n(m)\}_m$ such that

$$\lim_m (z_{m,n(m)}, w_{m,n(m)}) = (z, w) \quad \text{a.e. in } Q_T \times S_T, \quad (15)$$

$$\lim_m u_{m,n(m)} = u \quad \text{a.e. in } Q_T, \quad (16)$$

and

$$\lim_m u_{m,n(m)} = v \quad \text{a.e. on } S_T, \quad (17)$$

where $z(t, x), w(t, x), u(t, x), v(t, x) \in \bar{\mathbb{R}}$. Let us write

$$\begin{aligned} z_m &= z_{m,n(m)}, & w_m &= w_{m,n(m)}, \\ u_m &= u_{m,n(m)}, \\ f_m &= f_{m,n(m)}, & g_m &= g_{m,n(m)}, \\ z_{0m} &= z_{0m,n(m)} \quad \text{and} \quad w_{0m} = w_{0m,n(m)}. \end{aligned} \quad (18)$$

Step 2: Convergence of z_m, w_m . Let us see that

$$\lim_m (z_m, w_m) = (z, w) \quad \text{in } C([0, T]; X), \quad (19)$$

where $X = L^1(\Omega) \times L^1(\partial\Omega)$ provided with the natural norm

$$\|(f, g)\| := \|f\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}, \quad (f, g) \in X.$$

Observe that then $z(0) = z_0$ and $w(0) = w_0$ also hold.

Consider the operator $\mathcal{B}^{\gamma, \beta}$ defined in X by $(\hat{z}, \hat{w}) \in \mathcal{B}^{\gamma, \beta}(z, w)$ if and only if there exists $u \in W^{1,p}(\Omega)$ such that $z(x) \in \gamma(u(x))$ a.e. in Ω , $w(x) \in \beta(u(x))$ a.e. in $\partial\Omega$, and

$$\int_{\Omega} \mathbf{a}(x, Du) \cdot Dv \, dx = \int_{\Omega} \hat{z}v \, dx + \int_{\partial\Omega} \hat{w}v \, d\sigma \quad (20)$$

for all $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$. By results in [4] and [5], we know that the abstract Cauchy problem in X ,

$$\begin{cases} V'(t) + \mathcal{B}^{\gamma, \beta}(V(t)) \ni (f, g), & t \in (0, T), \\ V(0) = (z_0, w_0), \end{cases} \quad (21)$$

has a unique mild solution for any $f \in L^1(0, T; L^1(\Omega))$, $g \in L^1(0, T; L^1(\partial\Omega))$, $z_0 \in L^1(\Omega)$ and $w_0 \in L^1(\partial\Omega)$ satisfying (10) and (11). Moreover, under the hypothesis of Theorem 2.3, in [5] it is proved that the mild solution of problem (21) is the unique weak solution of $P_{\gamma, \beta}(f, g, z_0, w_0)$.

Therefore, (z_m, w_m) is the mild solution of problem (21) for data (f_m, g_m) and (z_{0m}, w_{0m}) . Since $(f_m, g_m) \rightarrow (f, g)$ in $L^1(0, T; X)$ and $(z_{0m}, w_{0m}) \rightarrow (z_0, w_0)$ in X , by the nonlinear semi-group theory, there exists $\lim_m (z_m, w_m)$ in $C([0, T]; X)$ and by (15), (19) holds, being (z, w) the mild solution of (21) for data (f, g) and (z_0, w_0) . We shall see that (z, w) is, in fact, a renormalized solution of problem $P_{\gamma, \beta}(f, g, z_0, w_0)$.

Step 3: Boundedness of $T_k(u_m)$. Let us see there exists $C_1 > 0$ such that, for any $k > 0$,

$$\int_0^T \int_{\Omega} |DT_k(u_m)|^p \, dx \, dt \leq \frac{k}{\lambda} (\|(f, g)\|_{L^1(0, T; X)} + \|(z_0, w_0)\|_X) \quad (22)$$

and

$$\|T_k((u_m(t))^\pm)\|_{L^p(\Omega)} \leq C_1(\|DT_k((u_m(t))^\pm)\|_{L^p(\Omega)} + 1) \quad \forall t \in [0, T]. \quad (23)$$

By Lemma 2.1, we have

$$\begin{aligned} & \int_0^T \int_\Omega \int_0^{z_m(t)} G((\gamma^{-1})^0(s)) ds \psi_t dx dt + \int_0^T \int_{\partial\Omega} \int_0^{w_m(t)} G((\beta^{-1})^0(s)) ds \psi_t d\sigma dt \\ &= \int_0^T \int_\Omega \mathbf{a}(x, Du_m) \cdot D(G(u_m)\psi) dx dt \\ & \quad - \int_0^T \int_\Omega f_m G(u_m) \psi dx dt - \int_0^T \int_{\partial\Omega} g_m G(u_m) \psi d\sigma dt \end{aligned} \quad (24)$$

for any bounded function of bounded variation $G(r)$ such that $G(u_m) \in L^p(0, T; W^{1,p}(\Omega))$ and for any $\psi \in \mathcal{D}([0, T] \times \mathbb{R}^N)$. Taking in (24) $\psi(t, x) = \varphi(t)$, $\varphi \in \mathcal{D}([0, T])$, and $G(r) = T_k(r)$, $k \geq 0$, we get

$$\begin{aligned} & \int_0^T \varphi_t \int_\Omega \int_0^{z_m(t)} T_k((\gamma^{-1})^0(s)) ds dx dt + \int_0^T \varphi_t \int_{\partial\Omega} \int_0^{w_m(t)} T_k((\beta^{-1})^0(s)) ds d\sigma dt \\ &= \int_0^T \varphi \int_\Omega \mathbf{a}(x, Du_m) \cdot DT_k(u_m) dx dt \\ & \quad - \int_0^T \varphi \int_\Omega f_m T_k(u_m) dx dt - \int_0^T \varphi \int_{\partial\Omega} g_m T_k(u_m) d\sigma dt. \end{aligned} \quad (25)$$

Therefore

$$\begin{aligned} & \frac{d}{dt} \int_\Omega \int_0^{z_m(t)} T_k((\gamma^{-1})^0(s)) ds dx + \frac{d}{dt} \int_{\partial\Omega} \int_0^{w_m(t)} T_k((\beta^{-1})^0(s)) ds d\sigma \\ & \quad + \int_\Omega \mathbf{a}(x, Du_m(t)) \cdot DT_k(u_m(t)) dx \\ &= \int_\Omega f_m(t) T_k(u_m(t)) dx + \int_{\partial\Omega} g_m(t) T_k(u_m(t)) d\sigma \end{aligned} \quad (26)$$

in $\mathcal{D}'([0, T])$. Integrating (26) from 0 to T and using (H_1) , we get (22).

In order to prove (23) we need to treat separately different cases. In the case $\mathcal{R}_{\gamma,\beta}^+ = +\infty$, let

$$M = \sup_{t \in [0, T]} \left(\int_{\Omega} z^+(t) dx + \int_{\partial\Omega} w^+(t) d\sigma \right) + 1.$$

Then, by (19) there exists $m_0 \in \mathbb{N}$ such that

$$\sup_{t \in [0, T]} \left(\int_{\Omega} (z_m)^+(t) dx + \int_{\partial\Omega} (w_m)^+(t) d\sigma \right) < M \quad \forall m \geq m_0.$$

In the case $\mathcal{R}_{\gamma,\beta}^+ < +\infty$, by (13) (see Remark 2.5(ii) and (iii)), there exist $M \in \mathbb{R}$ and $m_0 \in \mathbb{N}$ such that, for all $m \geq m_0$,

$$\sup_{t \in [0, T]} \left(\int_{\Omega} z_m(t) dx + \int_{\partial\Omega} w_m(t) d\sigma \right) < M < \mathcal{R}_{\gamma,\beta}^+.$$

Moreover, by (19),

$$\lim_{L \rightarrow +\infty} \left(\int_{\{x \in \Omega: z_m(t)(x) < -L\}} |z_m(t)| dx + \int_{\{x \in \partial\Omega: w_m(t)(x) < -L\}} |w_m(t)| d\sigma \right) = 0$$

uniformly in $m \in \mathbb{N}$ and $t \in [0, T]$.

Let us define

$$z_m^k = \begin{cases} z_m & \text{if } |u_m| < k, \\ \gamma^0(k) & \text{if } u_m \geq k, \\ \gamma^0(-k) & \text{if } u_m \leq -k, \end{cases}$$

and

$$w_m^k = \begin{cases} w_m & \text{if } |u_m| < k, \\ \beta^0(k) & \text{if } u_m \geq k, \\ \beta^0(-k) & \text{if } u_m \leq -k. \end{cases}$$

Then $z_m^k \in \gamma(T_k(u_m))$ a.e. in Q_T and $w_m^k \in \beta(T_k(u_m))$ a.e. in S_T . Now, in the case $\mathcal{R}_{\gamma,\beta}^+ = +\infty$, there exists $M \in \mathbb{R}$ such that, for all $k > 0$,

$$\sup_{t \in [0, T]} \left(\int_{\Omega} (z_m^k)^+(t) dx + \int_{\partial\Omega} (w_m^k)^+(t) d\sigma \right) < M \quad \forall m \geq m_0.$$

And in the case $\mathcal{R}_{\gamma,\beta}^+ < +\infty$, there exist $M < \mathcal{R}_{\gamma,\beta}^+$ and k_0 such that, for all $k \geq k_0$ and for all $m \geq m_0$,

$$\sup_{t \in [0, T]} \left(\int_{\Omega} z_m^k(t) dx + \int_{\partial\Omega} w_m^k(t) d\sigma \right) < M$$

and

$$\lim_{L \rightarrow +\infty} \left(\int_{\{x \in \Omega: z_m^k(t)(x) < -L\}} |z_m^k(t)| dx + \int_{\{x \in \partial\Omega: w_m^k(t)(x) < -L\}} |w_m^k(t)| d\sigma \right) = 0$$

uniformly in $m, k \in \mathbb{N}$ and $t \in [0, T]$. Therefore, by Lemma 3.1, (23) follows for the positive part of u_m . For the negative part of u_m we use again Lemma 3.1 for $\hat{u}_m = -u_m$, $\hat{z}_m = -z_m$, $\hat{w}_m = -w_m$ and the graphs $\hat{\gamma}(r) = -\gamma(-r)$ and $\hat{\beta}(r) = -\beta(-r)$.

Step 4: Convergence of $T_k(u_m)$. In this step we show that

$$u \text{ is finite a.e. in } Q_T, \quad (27)$$

$$u(t) \in \mathcal{T}_\tau^{1,p}(\Omega) \quad \text{a.e. } t \in]0, T[, \quad (28)$$

$$z \in \gamma(u) \quad \text{a.e. in } Q_T, \quad (29)$$

$$w \in \beta(u) \quad \text{a.e. in } S_T, \quad (30)$$

and, for any $k \in \mathbb{N}$,

$$\begin{aligned} T_k(u_m) \text{ converges to } T_k(u) \quad & \text{weakly in } L^p(0, T; W^{1,p}(\Omega)), \\ & \text{strongly in } L^p(0, T; L^p(\Omega)) \end{aligned} \quad (31)$$

and

$$T_k(u_m) \text{ converges to } T_k(u) \quad \text{in } L^p(0, T; L^p(\partial\Omega)). \quad (32)$$

Indeed, having in mind (22) and (23),

$$\begin{aligned} \mathcal{L}^{N+1}(\{(t, x) \in Q_T: u_m^\pm(t, x) \geq k\}) &\leq \int_0^T \int_\Omega \frac{|T_k((u_m(t))^\pm)|^p}{k^p} dx dt \\ &\leq \frac{C_2}{k^p} \int_0^T (1 + \|DT_k((u_m(t))^\pm)\|_{L^p(\Omega)}^p) dt \leq \frac{C_3}{k^p} (1 + k). \end{aligned}$$

This implies, taking limits first as m goes to $+\infty$ and after as k goes to $+\infty$, that (27) holds. Hence, again by (22), (31) and (32) hold for any $k > 0$, and consequently $u(t) \in \mathcal{T}^{1,p}(\Omega)$ a.e. $t \in]0, T[$.

Similarly, since

$$\begin{aligned}
& (\mathcal{L}^1 \times \mathcal{H}^{N-1}) (\{(t, x) \in S_T: u_m^\pm(t, x) \geq k\}) \\
& \leq \int_0^T \int_{\partial\Omega} \frac{|T_k((u_m)^\pm)|^p}{k^p} d\sigma dt \\
& \leq \frac{C_4}{k^p} \int_0^T (\|T_k((u_m)^\pm)\|_{L^p(\Omega)} + \|DT_k((u_m)^\pm)\|_{L^p(\Omega)})^p dt \leq \frac{C_5}{k^p} (1+k),
\end{aligned}$$

v is measurable in S_T , and (28) holds.

Finally, by (15)–(17), (27) and (28) and the facts that

$$\begin{aligned}
z_m & \in \gamma(u_m) \quad \text{a.e. in } Q_T, \\
w_m & \in \beta(u_m) \quad \text{a.e. in } S_T,
\end{aligned}$$

and γ and β are maximal monotone graphs, (29) and (30) hold.

Step 5: Uniform renormalized condition for u_m . Let us define

$$v(n) := \sup_m \int_{\{(t,x) \in Q_T: n < |u_m(t,x)| < n+1\}} \mathbf{a}(x, Du_m) \cdot Du_m dx dt.$$

Then

$$\lim_n v(n) = 0. \tag{33}$$

In order to prove (33) we take in (26) $k = n + 1$ and after $k = n$. Subtracting the corresponding equalities and integrating from 0 to T , we get

$$\begin{aligned}
0 & \leq \int_{\{(t,x) \in Q_T: n < |u_m(t,x)| < n+1\}} \mathbf{a}(x, Du_m) \cdot Du_m dx dt \\
& = - \int_{\Omega} \int_{z_{0m}}^{z_m(T)} G_n((\gamma^{-1})^0(s)) ds dx - \int_{\partial\Omega} \int_{w_{0m}}^{w_m(T)} G_n((\beta^{-1})^0(s)) ds d\sigma \\
& \quad + \int_0^T \int_{\Omega} f_m G_n(u_m) dx dt + \int_0^T \int_{\partial\Omega} g_m G_n(u_m) d\sigma dt,
\end{aligned} \tag{34}$$

where $G_n(r) := T_{n+1}(r) - T_n(r)$. Therefore, since

$$\lim_{n \rightarrow +\infty} \mathcal{L}^{N+1}(\{(t, x) \in Q_T: |u_m(t, x)| \geq n\}) = 0$$

and

$$\lim_{n \rightarrow +\infty} (\mathcal{L}^1 \times \mathcal{H}^{N-1})(\{(t, x) \in S_T: |u_m(t, x)| \geq n\}) = 0$$

uniformly in m , by equiintegrability, the two last terms on the right-hand side of equality (34) go to zero as n goes to $+\infty$. For the first term on the right-hand side of (34) we have

$$\begin{aligned} & - \int_{\Omega} \int_{z_{0m}}^{z_m(T)} G_n((\gamma^{-1})^0(s)) ds dx \\ & \leq \int_{\Omega} (z_0^+ - \sup \gamma(n))^+ dx + \int_{\Omega} (\inf \gamma(-n) - z_0^-)^- d\sigma, \end{aligned}$$

which converges to zero by (10). Similarly, we can handle with the second term on the right-hand side of (34) and the proof of (33) is concluded.

Step 6: Convergence of $\mathbf{a}(x, DT_k(u_m))$. Let us see that

$$\mathbf{a}(x, DT_k(u_m)) \rightharpoonup \mathbf{a}(x, DT_k(u)) \quad \text{weakly in } L^{p'}(Q_T) \text{ as } m \rightarrow +\infty. \quad (35)$$

Let $n \in \mathbb{N}$, $n > k$. Given any subsequence of u_m , by (22) and (H_2) , there exists a subsequence, still denoted by u_m , such that

$$\mathbf{a}(x, DT_k(u_m)) \rightharpoonup \Phi_k \quad \text{weakly in } (L^{p'}(Q_T))^N, \quad (36)$$

$$\mathbf{a}(x, DT_{n+1}(u_m)) \rightharpoonup \Phi_{n+1} \quad \text{weakly in } (L^{p'}(Q_T))^N \quad (37)$$

and

$$\mathbf{a}(x, DT_{n+1}(u_m)) \chi_{\{|u_m| > k\}} \rightharpoonup \Psi_{n+1,k} \quad \text{weakly in } (L^{p'}(Q_T))^N \quad (38)$$

as $m \rightarrow +\infty$. Let us prove that, for any $\varphi \in \mathcal{D}([0, T])$, $0 \leq \varphi \leq 1$,

$$\lim_{m \rightarrow +\infty} \int_0^T \int_{\Omega} \varphi \mathbf{a}(x, Du_m) \cdot DT_k(u_m) dx dt \leq \int_0^T \int_{\Omega} \varphi \Phi_k \cdot DT_k(u) dx dt. \quad (39)$$

Then, by Minty–Browder’s method, it is easy to see that

$$\Phi_k = \mathbf{a}(x, DT_k(u)), \quad (40)$$

and (35) is proved.

Now, in order to get (39), we take limit in (25) to obtain

$$\begin{aligned}
& \lim_{m \rightarrow +\infty} \int_0^T \varphi \int_{\Omega} \mathbf{a}(x, Du_m) \cdot DT_k(u_m) dx dt \\
&= \int_0^T \varphi_t \int_{\Omega} \int_0^{z(t)} T_k((\gamma^{-1})^0(s)) ds dx dt + \int_0^T \varphi_t \int_{\partial\Omega} \int_0^{w(t)} T_k((\beta^{-1})^0(s)) ds d\sigma dt \\
&+ \int_0^T \varphi \int_{\Omega} f T_k(u) dx dt + \int_0^T \varphi \int_{\partial\Omega} g T_k(u) d\sigma dt.
\end{aligned} \tag{41}$$

Consequently, it is enough to prove that

$$\begin{aligned}
& \int_0^T \varphi \int_{\Omega} \Phi_k \cdot DT_k(u) dx dt \\
&\geq \int_0^T \varphi_t \int_{\Omega} \int_0^{z(t)} T_k((\gamma^{-1})^0(s)) ds dx dt + \int_0^T \varphi_t \int_{\partial\Omega} \int_0^{w(t)} T_k((\beta^{-1})^0(s)) ds d\sigma dt \\
&+ \int_0^T \varphi \int_{\Omega} f T_k(u) dx dt + \int_0^T \varphi \int_{\partial\Omega} g T_k(u) d\sigma dt.
\end{aligned} \tag{42}$$

To this end we use the regularization method of Landes [39]. For $k, v \in \mathbb{N}$, we define the regularization in time of the function $T_k(u)$ given by

$$(T_k(u))_v(t, x) := v \int_{-\infty}^t e^{v(s-t)} T_k(u(s, x)) ds,$$

extending $T_k(u)$ by 0 for $s < 0$. Observe that $(T_k(u))_v \in L^p(0, T; W^{1,p}(\Omega)) \cap L^\infty(Q)$, it is differentiable for a.e. $t \in (0, T)$ with

$$\begin{aligned}
& |(T_k(u))_v(t, x)| \leq k(1 - e^{-vt}) < k \quad \text{a.e.}, \\
& \frac{\partial (T_k(u))_v}{\partial t} = v(T_k(u) - (T_k(u))_v) \in L^p(0, T; W^{1,p}(\Omega)) \cap L^\infty(Q),
\end{aligned}$$

$$(T_k(u))_v(0, x) = 0 \quad \forall x \in \Omega,$$

$$\lim_{v \rightarrow \infty} (T_k(u))_v = T_k(u) \quad \text{in } L^p(0, T; W^{1,p}(\Omega)) \text{ and in } L^p([0, T] \times \partial\Omega)$$

and, moreover,

$$\int_0^T \varphi \int_{\Omega} \Phi_k \cdot DT_k(u) dx dt = \lim_{v \rightarrow \infty} \lim_{m \rightarrow \infty} \int_0^T \varphi \int_{\Omega} \mathbf{a}(x, DT_k(u_m)) \cdot D(T_k(u))_v dx dt.$$

By considering $H_n(r) = \inf(1, (n+1-|r|)^+)$, for any $n > k$, which implies $H_n(u_m) = 1$ if $|u_m| \leq k$, we have

$$\begin{aligned}
 & \int_0^T \varphi \int_{\Omega} \Phi_k \cdot DT_k(u) \, dx \, dt \\
 &= \lim_{v \rightarrow \infty} \lim_{m \rightarrow \infty} \int_0^T \varphi \int_{\Omega} \mathbf{a}(x, DT_k(u_m)) \cdot D(H_n(u_m)(T_k(u)))_v \, dx \, dt \\
 &= \lim_{v \rightarrow \infty} \lim_{m \rightarrow \infty} \left(\int_0^T \varphi \int_{\Omega} \mathbf{a}(x, DT_{n+1}(u_m)) \cdot D(H_n(u_m)(T_k(u)))_v \, dx \, dt \right. \\
 &\quad \left. - \int_{\{(t,x) \in Q_T: k < |u_m(t,x)| \leq n+1\}} \varphi \mathbf{a}(x, DT_{n+1}(u_m)) \cdot D(H_n(u_m)(T_k(u)))_v \, dx \, dt \right) \\
 &= \lim_{v \rightarrow \infty} \lim_{m \rightarrow \infty} \left(\int_0^T \varphi \int_{\Omega} \mathbf{a}(x, DT_{n+1}(u_m)) \cdot D(H_n(u_m)(T_k(u)))_v \, dx \, dt \right. \\
 &\quad \left. - \int_{\{(t,x) \in Q_T: k < |u_m(t,x)| \leq n+1\}} \varphi \mathbf{a}(x, Du_m) \cdot D((T_k(u))_v) H_n(u_m) \, dx \, dt \right. \\
 &\quad \left. - \int_{\{(t,x) \in Q_T: k < |u_m(t,x)| \leq n+1\}} \varphi \mathbf{a}(x, Du_m) \cdot Du_m H'_n(u_m)(T_k(u))_v \, dx \, dt \right) \\
 &= \lim_{v \rightarrow \infty} \lim_{m \rightarrow \infty} \left(\int_0^T \varphi \int_{\Omega} \mathbf{a}(x, DT_{n+1}(u_m)) \cdot D(H_n(u_m)(T_k(u)))_v \, dx \, dt \right. \\
 &\quad \left. - \int_0^T \varphi \int_{\Omega} \mathbf{a}(x, DT_{n+1}(u_m)) \cdot D((T_k(u))_v) H_n(u_m) \chi_{\{(t,x) \in Q_T: |u_m(t,x)| > k\}} \, dx \, dt \right. \\
 &\quad \left. - \int_{\{(t,x) \in Q_T: k < |u_m(t,x)| \leq n+1\}} \varphi \mathbf{a}(x, Du_m) \cdot Du_m H'_n(u_m)(T_k(u))_v \, dx \, dt \right).
 \end{aligned}$$

Since $|(T_k(u))_v| = k(1 - e^{-v})$ in $\{(t, x) \in Q_T: |u(t, x)| \geq k\}$, having in mind (38), we get

$$\begin{aligned}
& \lim_{v \rightarrow \infty} \lim_{m \rightarrow \infty} \int_0^T \varphi \int_{\Omega} \mathbf{a}(x, DT_{n+1}(u_m)) \cdot D(T_k(u))_v H_n(u_m) \chi_{\{(t,x) \in Q_T: |u_m(t,x)| > k\}} dx dt \\
&= \lim_{v \rightarrow \infty} \lim_{m \rightarrow \infty} \int_0^T \varphi \left[\int_{\Omega} \mathbf{a}(x, DT_{n+1}(u_m)) \cdot D(T_k(u))_v \right. \\
&\quad \left. \times H_n(u_m) \chi_{\{(t,x) \in Q_T: |u_m(t,x)| > k\}} \chi_{\{(t,x) \in Q_T: |u(t,x)| < k\}} dx \right] dt = 0.
\end{aligned}$$

Hence, for any $n > k$, we have

$$\begin{aligned}
& \int_0^T \varphi \int_{\Omega} \Phi_k \cdot DT_k(u) dx dt \\
&= \lim_{v \rightarrow \infty} \lim_{m \rightarrow \infty} \left(\int_0^T \varphi \int_{\Omega} \mathbf{a}(x, DT_{n+1}(u_m)) \cdot D(H_n(u_m)(T_k(u)))_v dx dt \right. \\
&\quad \left. - \int_{\{(t,x) \in Q_T: k < |u_m(t,x)| \leq n+1\}} \varphi \mathbf{a}(x, Du_m) \cdot Du_m H'_n(u_m)(T_k(u))_v dx dt \right). \quad (43)
\end{aligned}$$

Now,

$$\left| \int_{\{(t,x) \in Q_T: k < |u_m(t,x)| \leq n+1\}} \varphi \mathbf{a}(x, Du_m) \cdot Du_m H'_n(u_m)(T_k(u))_v dx dt \right| \leq kv(n),$$

thus

$$- \int_{\{(t,x): k < |u_m(t,x)| \leq n+1\}} \varphi \mathbf{a}(x, Du_m) \cdot Du_m H'_n(u_m)(T_k(u))_v dx dt \geq -kv(n),$$

so that, by (43) and (33), we get

$$\begin{aligned}
& \int_0^T \varphi \int_{\Omega} \Phi_k \cdot DT_k(u) dx dt \\
&\geq \liminf_{n \rightarrow \infty} \liminf_{v \rightarrow \infty} \liminf_{m \rightarrow \infty} \int_0^T \varphi \int_{\Omega} \mathbf{a}(x, DT_{n+1}(u_m)) \cdot D(H_n(u_m)(T_k(u)))_v dx dt. \quad (44)
\end{aligned}$$

Since H_n is a bounded function of bounded variation, from (24), by approximation of $(T_k(u))_v \varphi$, we deduce

$$\begin{aligned}
& \int_0^T \varphi \int_{\Omega} \mathbf{a}(x, DT_{n+1}(u_m)) \cdot D(H_n(u_m)(T_k(u))_v) dx dt \\
&= \int_0^T \int_{\Omega} \mathbf{a}(x, Du_m) \cdot D(H_n(u_m)(T_k(u))_v) \varphi dx dt \\
&= \int_0^T \int_{\Omega} b_n^\gamma(z_m)(\varphi(T_k(u))_v)_t dx dt + \int_0^T \int_{\partial\Omega} b_n^\beta(w_m)(\varphi(T_k(u))_v)_t d\sigma dt \\
&\quad + \int_0^T \int_{\Omega} f_m H_n(u_m) \varphi(T_k(u))_v dx dt + \int_0^T \int_{\partial\Omega} g_m H_n(u_m) \varphi(T_k(u))_v d\sigma dt, \tag{45}
\end{aligned}$$

where

$$b_n^\gamma(r) = \int_0^r H_n((\gamma^{-1})^0(s)) ds \quad \text{for } r \in \text{Ran}(\gamma), \tag{46}$$

and

$$b_n^\beta(r) = \int_0^r H_n((\beta^{-1})^0(s)) ds \quad \text{for } r \in \text{Ran}(\beta). \tag{47}$$

Letting $m \rightarrow \infty$ in (45), we have

$$\begin{aligned}
& \lim_{m \rightarrow \infty} \int_0^T \int_{\Omega} \mathbf{a}(x, Du_m) \cdot D(H_n(u_m)(T_k(u))_v) \varphi dx dt \\
&= \int_0^T \int_{\Omega} b_n^\gamma(z)(\varphi(T_k(u))_v)_t dx dt + \int_0^T \int_{\partial\Omega} b_n^\beta(w)(\varphi(T_k(u))_v)_t d\sigma dt \\
&\quad + \int_0^T \int_{\Omega} f H_n(u) \varphi(T_k(u))_v dx dt + \int_0^T \int_{\partial\Omega} g H_n(u) \varphi(T_k(u))_v d\sigma dt.
\end{aligned}$$

For the first term on the right-hand side, using the fact that $(T_k(u))_v = T_k((T_k(u))_v)$, $z \in \gamma(u)$, the monotonicity of b_n^γ and the integration by parts formula, we get

$$\begin{aligned}
& \int_0^T \int_{\Omega} b_n^\gamma(z) (\varphi(T_k(u))_v)_t dx dt \\
&= \int_0^T \int_{\Omega} b_n^\gamma(z) \varphi_t(T_k(u))_v dx dt + \nu \int_0^T \int_{\Omega} b_n^\gamma(z) \varphi(T_k(u) - (T_k(u))_v) dx dt \\
&\geq \int_0^T \int_{\Omega} b_n^\gamma(z) \varphi_t(T_k(u))_v dx dt + \nu \int_0^T \int_{\Omega} b_n^\gamma(\gamma^0((T_k(u))_v)) \varphi(T_k(u) - (T_k(u))_v) dx dt \\
&= \int_0^T \int_{\Omega} b_n^\gamma(z) \varphi_t(T_k(u))_v dx dt + \int_0^T \int_{\Omega} b_n^\gamma(\gamma^0((T_k(u))_v)) ((T_k(u))_v)_t \varphi dx dt \\
&= \int_0^T \int_{\Omega} b_n^\gamma(z) \varphi_t(T_k(u))_v dx dt - \int_0^T \int_{\Omega} \int_0^{(T_k(u))_v} b_n^\gamma(\gamma^0(s)) ds \varphi_t dx dt.
\end{aligned}$$

Now, letting $\nu \rightarrow \infty$, we deduce that

$$\begin{aligned}
& \liminf_{\nu \rightarrow \infty} \int_0^T \int_{\Omega} b_n^\gamma(z) (\varphi(T_k(u))_v)_t dx dt \\
&\geq \int_0^T \int_{\Omega} b_n^\gamma(z) \varphi_t T_k(u) dx dt - \int_0^T \int_{\Omega} \int_0^{T_k(u)} b_n^\gamma(\gamma^0(s)) ds \varphi_t dx dt,
\end{aligned}$$

so that,

$$\begin{aligned}
& \liminf_{n \rightarrow \infty} \liminf_{\nu \rightarrow \infty} \liminf_{m \rightarrow \infty} \int_0^T \int_{\Omega} b_n^\gamma(z_m) (\varphi(T_k(u))_v)_t dx dt \\
&\geq \int_0^T \int_{\Omega} \varphi_t z T_k(u) dx dt - \int_0^T \int_{\Omega} \int_0^{T_k(u)} \gamma^0(s) ds \varphi_t dx dt.
\end{aligned}$$

Using the fact that, since $z \in \gamma(u)$,

$$z T_k(u) - \int_0^{T_k(u)} \gamma^0(s) ds = \int_0^z T_k((\gamma^{-1})^0(s)) ds,$$

we obtain that

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \liminf_{v \rightarrow \infty} \liminf_{m \rightarrow \infty} \int_0^T \int_{\Omega} b_n^\gamma(z_m)(\varphi(T_k(u))_v)_t dx dt \\ & \geq \int_0^T \varphi_t \int_{\Omega} \int_0^{z(t)} T_k((\gamma^{-1})^0(s)) ds dx dt. \end{aligned}$$

In the same way, we get that

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \liminf_{v \rightarrow \infty} \liminf_{m \rightarrow \infty} \int_0^T \int_{\partial\Omega} b_n^\beta(w_m)(\varphi(T_k(u))_v)_t d\sigma dt \\ & \geq \int_0^T \varphi_t \int_{\partial\Omega} \left(\int_0^{w(t)} T_k((\beta^{-1})^0(s)) ds \right) d\sigma dt. \end{aligned}$$

Then, passing to the limit in (45), by (44), (42) follows.

Step 7: Passing to the limit. In this step we see that

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left(\int_0^{z(t)} H((\gamma^{-1})^0(s)) ds \right) \xi dx + \frac{d}{dt} \int_{\partial\Omega} \left(\int_0^{w(t)} H((\beta^{-1})^0(s)) ds \right) \xi d\sigma \\ & \quad + \int_{\Omega} \mathbf{a}(x, Du(t)) \cdot D(H(u(t))\xi) dx \\ & = \int_{\Omega} f(t)H(u(t))\xi dx + \int_{\partial\Omega} g(t)H(u(t))\xi d\sigma \end{aligned} \quad (48)$$

in $\mathcal{D}'([0, T])$.

By Step 6, for any $\varphi \in \mathcal{D}([0, T])$, $0 \leq \varphi \leq 1$, we have

$$\lim_{m \rightarrow +\infty} \int_0^T \varphi \int_{\Omega} (\mathbf{a}(x, DT_k(u_m)) - \mathbf{a}(x, DT_k(u))) \cdot (DT_k(u_m) - DT_k(u)) dx dt = 0.$$

Then, we can suppose, extracting a subsequence if necessary, that

$$\varphi(\mathbf{a}(x, DT_k(u_m)) - \mathbf{a}(x, DT_k(u))) \cdot (DT_k(u_m) - DT_k(u)) \rightarrow 0 \quad (49)$$

in $L^1(Q_T)$, a.e. in Q_T , and is dominated in $L^1(Q_T)$.

Taking in (24) $G(r) = H(r)$, being $H: \mathbb{R} \rightarrow \mathbb{R}$ a Lipschitz continuous function of compact support, and $\psi(t, x) = \varphi(t)\xi(x)$, $\varphi \in \mathcal{D}([0, T])$ and $\xi \in C^1(\bar{\Omega})$, we have

$$\begin{aligned}
& \int_0^T \int_{\Omega} \left(\int_0^{z_m(t)} H((\gamma^{-1})^0(s)) ds \right) \xi \varphi_t dx dt + \int_0^T \int_{\partial\Omega} \left(\int_0^{w_m(t)} H((\beta^{-1})^0(s)) ds \right) \xi \varphi_t d\sigma dt \\
&= \int_0^T \varphi \int_{\Omega} \mathbf{a}(x, Du_m) \cdot D(H(u_m)\xi) dx dt \\
&\quad - \int_0^T \varphi \int_{\Omega} f_m H(u_m) \xi dx dt - \int_0^T \varphi \int_{\partial\Omega} g_m(t) H(u_m) \xi d\sigma dt.
\end{aligned} \tag{50}$$

Now, if $\text{supp}(H) \subset [-M, M]$,

$$\begin{aligned}
& \int_0^T \varphi \int_{\Omega} \mathbf{a}(x, Du_m) \cdot D(H(u_m)\xi) dx dt \\
&= \int_0^T \varphi \int_{\Omega} \mathbf{a}(x, DT_M(u_m)) \cdot D(H(T_M(u_m))\xi) dx dt \\
&= \int_0^T \varphi \int_{\Omega} H(T_M(u_m)) \mathbf{a}(x, DT_M(u_m)) \cdot D\xi dx dt \\
&\quad + \int_0^T \varphi \int_{\Omega} \xi H'(T_M(u_m)) \mathbf{a}(x, DT_M(u_m)) \cdot DT_M(u_m) dx dt \\
&= \int_0^T \varphi \int_{\Omega} H(T_M(u_m)) \mathbf{a}(x, DT_M(u_m)) \cdot D\xi dx dt \\
&\quad + \int_0^T \varphi \left[\int_{\Omega} \xi H'(T_M(u_m)) (\mathbf{a}(x, DT_M(u_m)) - \mathbf{a}(x, DT_M(u))) \right. \\
&\quad \left. \times (DT_M(u_m) - DT_M(u)) dx \right] dt \\
&\quad + \int_0^T \varphi \int_{\Omega} \xi H'(T_M(u_m)) \mathbf{a}(x, DT_M(u_m)) \cdot DT_M(u) dx dt \\
&\quad + \int_0^T \varphi \int_{\Omega} \xi H'(T_M(u_m)) \mathbf{a}(x, DT_M(u)) \cdot DT_M(u_m) dx dt \\
&\quad - \int_0^T \varphi \int_{\Omega} \xi H'(T_M(u_m)) \mathbf{a}(x, DT_M(u)) \cdot DT_M(u) dx dt.
\end{aligned}$$

Since by approximation we can assume H to be smooth, by (35) and (49), we get

$$\begin{aligned}
 & \lim_{m \rightarrow \infty} \int_0^T \varphi \int_{\Omega} \mathbf{a}(x, Du_m) \cdot D(H(u_m)\xi) \, dx \, dt \\
 &= \int_0^T \varphi \int_{\Omega} H(T_M(u)) \mathbf{a}(x, DT_M(u)) \cdot D\xi \, dx \, dt \\
 &\quad + \int_0^T \varphi \int_{\Omega} \xi H'(T_M(u)) \mathbf{a}(x, DT_M(u)) \cdot DT_M(u) \, dx \, dt \\
 &= \int_0^T \varphi \int_{\Omega} \mathbf{a}(x, Du) \cdot D(H(u)\xi) \, dx \, dt.
 \end{aligned}$$

Consequently, taking limit in (50) as $m \rightarrow \infty$, (48) follows.

Step 8: Renormalized condition. Let us see finally that

$$\lim_{n \rightarrow +\infty} \int_{\{(t,x) \in Q_T : n \leq |u(t,x)| \leq n+1\}} \mathbf{a}(x, Du) \cdot Du \, dx \, dt = 0. \quad (51)$$

By (49), we have

$$\begin{aligned}
 & \lim_{m \rightarrow +\infty} \int_s^{T-s} \int_{\Omega} \mathbf{a}(x, DT_k(u_m)) \cdot DT_k(u_m) \, dx \, dt \\
 &= \int_s^{T-s} \int_{\Omega} \mathbf{a}(x, DT_k(u)) \cdot DT_k(u) \, dx \, dt
 \end{aligned} \quad (52)$$

for any $0 < s < T/2$. Taking now in (52) $k = n+1$, $k = n$ and subtracting the resulting equalities, for any $0 < s < T/2$,

$$\begin{aligned}
 & \lim_m \int_s^{T-s} \int_{\{x \in \Omega : n \leq |u_m(t,x)| \leq n+1\}} \mathbf{a}(x, Du_m(t)) \cdot Du_m(t) \, dx \, dt \\
 &= \int_s^{T-s} \int_{\{x \in \Omega : n \leq |u(t,x)| \leq n+1\}} \mathbf{a}(x, Du(t)) \cdot Du(t) \, dx \, dt.
 \end{aligned}$$

Then, by the definition of $v(n)$,

$$\int_s^{T-s} \int_{\{x \in \Omega: n \leq |u(t,x)| \leq n+1\}} \mathbf{a}(x, Du(t)) \cdot Du(t) dx dt \leq v(n).$$

Therefore, taking limits as s goes to 0, and taking into account (33), (51) is proved.

With this last step the proof of the existence part of Theorem 2.6 is concluded. \square

Remark 3.2. Using (49), we can get, as in [18, Lemma 5], the strong convergence of $\{DT_k(u_m)\}_m$.

4. Uniqueness of renormalized solution

In this section we prove the uniqueness part of Theorem 2.6 using as main tool the concept of *integral solution* due to Ph. B enilan (see [12,14]).

Definition 4.1. A function $V = (z, w) \in C([0, T]; X)$ is an *integral solution* of (21) in $[0, T]$, if for every $(\hat{f}, \hat{g}) \in \mathcal{B}^{\gamma, \beta}(\hat{z}, \hat{w})$ we have

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |z(t) - \hat{z}| dx + \frac{d}{dt} \int_{\partial\Omega} |w(t) - \hat{w}| d\sigma \\ & \leq \int_{\Omega} (f(t) - \hat{f}) \operatorname{sign}_0(z(t) - \hat{z}) dx + \int_{\{x \in \Omega: z(t,x) = \hat{z}(x)\}} |f(t) - \hat{f}| dx \\ & \quad + \int_{\partial\Omega} (g(t) - \hat{g}) \operatorname{sign}_0(w(t) - \hat{w}) d\sigma + \int_{\{x \in \partial\Omega: w(t,x) = \hat{w}(x)\}} |g(t) - \hat{g}| d\sigma \end{aligned}$$

in $\mathcal{D}'(]0, T[)$, and $V(0) = (z_0, w_0)$.

Under the hypothesis $\operatorname{Dom}(\gamma) = \mathbb{R}$ and either $\operatorname{Dom}(\beta) = \mathbb{R}$ or \mathbf{a} smooth, the operator $\mathcal{B}^{\gamma, \beta}$ (see Section 3) is accretive in X (see [4] and [5]). In [5, Theorem 3.6] the existence of mild solutions of problem (21) is proved under conditions (11) and (10). Now, mild solutions and integral solutions of problem (21) coincide (see [12,14]). In Theorem 4.3, we shall prove that a renormalized solution of $P_{\gamma, \beta}(f, g, z_0, w_0)$ in $[0, T]$ is an integral solution of (21). Consequently, since in fact $\mathcal{B}^{\gamma, \beta}$ is T -accretive in X (see [4] and [5]), the contraction principle (12) follows by the nonlinear semigroup theory. Finally, under the assumptions of Theorem 2.6(i), the mild solution of (21) in $[0, T]$ is the unique renormalized solution of $P_{\gamma, \beta}(f, g, z_0, w_0)$ in $[0, T]$.

We shall use the following integration by parts formula.

Lemma 4.2. Let (z, w) be a renormalized solution of $P_{\gamma, \beta}(f, g, z_0, w_0)$ in $[0, T]$. Let $k > 0$, $n \in \mathbb{N}$, $H_n(r) = \inf(1, (n+1 - |r|)^+)$, and $h \in W^{1,p}(\Omega)$. Then,

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} \left(\int_0^{z(t)} H_n((\gamma^{-1})^0(s)) T_k((\gamma^{-1})^0(s) - h) ds \right) \psi dx \\
& + \frac{d}{dt} \int_{\partial\Omega} \left(\int_0^{w(t)} H_n((\beta^{-1})^0(s)) T_k((\beta^{-1})^0(s) - h) ds \right) \psi d\sigma \\
& + \int_{\Omega} \mathbf{a}(x, Du(t)) D(H_n(u(t)) T_k(u(t) - h) \psi) dx \\
& = \int_{\Omega} f(t) H_n(u(t)) T_k(u(t) - h) \psi dx + \int_{\partial\Omega} g(t) H_n(u(t)) T_k(u(t) - h) \psi d\sigma
\end{aligned}$$

in $\mathcal{D}'([0, T])$, for any $\psi \in \mathcal{D}(\mathbb{R}^N)$, being u the function given in the definition of (z, w) as renormalized solution.

Proof. Let b_n^γ and b_n^β be defined as in (46) and (47), respectively. Since (z, w) is a renormalized solution of $P_{\gamma, \beta}(f, g, z_0, w_0)$ in $[0, T]$, for $\xi \in C^1(\bar{\Omega})$,

$$\begin{aligned}
& -\frac{d}{dt} \int_{\Omega} b_n^\gamma(z(t)) \xi dx - \frac{d}{dt} \int_{\partial\Omega} b_n^\beta(w(t)) \xi d\sigma \\
& = \int_{\Omega} \mathbf{a}(x, Du(t)) \cdot D(H_n(u(t)) \xi) dx - \int_{\Omega} f(t) H_n(u(t)) \xi dx - \int_{\partial\Omega} g(t) H_n(u(t)) \xi d\sigma
\end{aligned}$$

in $\mathcal{D}'([0, T])$. Therefore, since $b_n^\gamma(z(t)) \in (b_n^\gamma \circ \gamma)(u(t))$, $b_n^\beta(z(t)) \in (b_n^\beta \circ \beta)(u(t))$, by Lemma 2.1, applied with $H(x, r) = T_k(r - T_m(h))$ and

$$(F(t), \xi) = \int_{\Omega} \mathbf{a}(x, T_{k+m+n}(u(t))) D\xi dx,$$

$m \in \mathbb{N}$, it follows that

$$\begin{aligned}
& \int_0^T \int_{\Omega} \left(\int_0^{b_n^\gamma(z(t))} T_k(((b_n^\gamma \circ \gamma)^{-1})^0(s) - T_m(h)) ds \right) \psi_t dx dt \\
& + \int_0^T \int_{\partial\Omega} \left(\int_0^{b_n^\beta(w(t))} T_k(((b_n^\beta \circ \beta)^{-1})^0(s) - T_m(h)) ds \right) \psi_t d\sigma dt \\
& = \int_0^T \int_{\Omega} \mathbf{a}(x, Du) \cdot D(H_n(u) T_k(u - T_m(h)) \psi) dx dt
\end{aligned}$$

$$\begin{aligned}
& - \int_0^T \int_{\Omega} f H_n(u) T_k(u - T_m(h)) \psi \, dx \, dt \\
& - \int_0^T \int_{\partial\Omega} g H_n(u) T_k(u - T_m(h)) \psi \, d\sigma \, dt
\end{aligned}$$

for any $\psi \in \mathcal{D}([0, T[\times \mathbb{R}^N)$. Therefore, by the change of variables formula,

$$\begin{aligned}
& \int_0^T \int_{\Omega} \left(\int_0^{z(t)} H_n((\gamma^{-1})^0(s)) T_k(((b_n^\gamma \circ \gamma)^{-1})^0(b_n^\gamma(s)) - T_m(h)) \, ds \right) \psi_t \, dx \, dt \\
& + \int_0^T \int_{\partial\Omega} \left(\int_0^{w(t)} H_n((\beta^{-1})^0(s)) T_k(((b_n^\beta \circ \beta)^{-1})^0(b_n^\beta(s)) - T_m(h)) \, ds \right) \psi_t \, d\sigma \, dt \\
& = \int_0^T \int_{\Omega} \mathbf{a}(x, Du) \cdot D(H_n(u) T_k(u - T_m(h)) \psi) \, dx \, dt \\
& - \int_0^T \int_{\Omega} f H_n(u) T_k(u - T_m(h)) \psi \, dx \, dt - \int_0^T \int_{\partial\Omega} g H_n(u) T_k(u - T_m(h)) \psi \, d\sigma \, dt \quad (53)
\end{aligned}$$

for any $\psi \in \mathcal{D}([0, T[\times \mathbb{R}^N)$. Observe that

$$\begin{aligned}
& \int_0^{z(t)} H_n((\gamma^{-1})^0(s)) T_k(((b_n^\gamma \circ \gamma)^{-1})^0(b_n^\gamma(s)) - T_m(h)) \, ds \\
& = \int_0^{z(t)} H_n((\gamma^{-1})^0(s)) T_k((\gamma^{-1})^0(s) - T_m(h)) \, ds
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^{w(t)} H_n((\beta^{-1})^0(s)) T_k(((b_n^\beta \circ \beta)^{-1})^0(b_n^\beta(s)) - T_m(h)) \, ds \\
& = \int_0^{w(t)} H_n((\beta^{-1})^0(s)) T_k((\beta^{-1})^0(s) - T_m(h)) \, ds.
\end{aligned}$$

Indeed, let us see, for example, that

$$\begin{aligned}
& H_n((\gamma^{-1})^0(s)) T_k(((b_n^\gamma \circ \gamma)^{-1})^0(b_n^\gamma(s)) - T_m(h)) \\
&= H_n((\gamma^{-1})^0(s)) T_k((\gamma^{-1})^0(s) - T_m(h)).
\end{aligned} \tag{54}$$

If $s = 0$ then $(\gamma^{-1})^0(s) = 0 = ((b_n^\gamma \circ \gamma)^{-1})^0(b_n^\gamma(s))$ and (54) holds. If $b_n^\gamma(s) = 0$ and $s \neq 0$ then $H_n((\gamma^{-1})^0(s)) = 0$ and (54) also holds. If $b_n^\gamma(s) > 0$ then $0 \leq (\gamma^{-1})^0(s) \in (b_n^\gamma \circ \gamma)^{-1}(b_n^\gamma(s))$, and if $\alpha \in (b_n^\gamma \circ \gamma)^{-1}(b_n^\gamma(s))$ then there exists $c \in \gamma(\alpha)$ such that $b_n^\gamma(s) = b_n^\gamma(c)$; now, if $s \leq c$ it is easy to see that $(\gamma^{-1})^0(s) \leq \alpha$, so $(\gamma^{-1})^0(s) = ((b_n^\gamma \circ \gamma)^{-1})^0(b_n^\gamma(s))$, and, if $s > c$ then $H_n((\gamma^{-1})^0(s)) = 0$, therefore in any case (54) holds. Similarly, if $b_n^\gamma(s) < 0$, (54) is true.

Therefore, taking limit as m goes to $+\infty$ in (53) we finish the proof. \square

To prove the following theorem we use a similar scheme to that used in the proof of Theorem 5.3 in [5]. Now here, we have to overcome the added difficulties due to the fact that for u we only know that its truncations are in $L^p(0, T; W^{1,p}(\Omega))$. In this sense the renormalized condition (6) plays a role.

Theorem 4.3. *Let (z, w) be a renormalized solution of $P_{\gamma, \beta}(f, g, z_0, w_0)$ in $[0, T]$. Let $(\hat{f}, \hat{g}) \in \mathcal{B}^{\gamma, \beta}(\hat{z}, \hat{w})$. Then,*

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} |z(t) - \hat{z}| dx + \frac{d}{dt} \int_{\partial\Omega} |w(t) - \hat{w}| d\sigma \\
& \leq \int_{\Omega} (f(t) - \hat{f}) \operatorname{sign}_0(z(t) - \hat{z}) dx + \int_{\{x \in \Omega: z(t, x) = \hat{z}(x)\}} |f(t) - \hat{f}| dx \\
& + \int_{\partial\Omega} (g(t) - \hat{g}) \operatorname{sign}_0(w(t) - \hat{w}) d\sigma + \int_{\{x \in \partial\Omega: w(t, x) = \hat{w}(x)\}} |g(t) - \hat{g}| d\sigma
\end{aligned}$$

in $\mathcal{D}'([0, T])$, that is, since $(z(0), w(0)) = (z_0, w_0)$, (z, w) is an integral solution of (21) in $[0, T]$.

Proof. We divide the proof in three steps.

Step 1: Inequality inside Ω . Let H_n be as in Lemma 4.2, $\psi \in \mathcal{D}(\Omega)$, $0 \leq \psi \leq 1$, $\rho \in W^{1,p}(\Omega)$, $-1 \leq \rho \leq 1$. Given $(\hat{f}, \hat{g}) \in \mathcal{B}^{\gamma, \beta}(\hat{z}, \hat{w})$ there exists $\hat{u} \in W^{1,p}(\Omega)$ such that $\hat{z}(x) \in \gamma(\hat{u}(x))$ a.e. in Ω , $\hat{w}(x) \in \beta(\hat{u}(x))$ a.e. in $\partial\Omega$, and

$$\int_{\Omega} \mathbf{a}(x, D\hat{u}) \cdot Dv dx = \int_{\Omega} \hat{f} v dx + \int_{\partial\Omega} \hat{g} v d\sigma$$

for all $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$. Then, if u is the function given in the definition of (z, w) as renormalized solution, we have, for $0 < \hat{t} \leq t < T$,

$$\begin{aligned}
& \int_{\Omega} \int_{\hat{z}}^{z(t)} H_n((\gamma^{-1})^0(s)) \operatorname{sign}_0(s - \hat{z}) \psi \, ds \, dx \\
& + \int_{\Omega} \int_{\hat{z}(\hat{t})}^{z(t)} H_n((\gamma^{-1})^0(s)) (\rho - \operatorname{sign}_0(s - \hat{z})) \chi_{\{s: (\gamma^{-1})^0(s) = \hat{u}\}} \psi \, ds \, dx \\
& + \int_{\hat{t}}^t \int_{\Omega} (\mathbf{a}(x, Du(s)) - \mathbf{a}(x, D\hat{u})) \cdot Du(s) H'_n(u(s)) \operatorname{sign}_0(u(s) - \hat{u}) \psi \, dx \, ds \\
& + \int_{\hat{t}}^t \int_{\Omega} (\mathbf{a}(x, Du(s)) - \mathbf{a}(x, D\hat{u})) \cdot D\psi H_n(u(s)) \operatorname{sign}_0(u(s) - \hat{u}) \, dx \, ds \\
& \leq \int_{\hat{t}}^t \left[\int_{\Omega} (f(s) - \hat{f}) H_n(u(s)) \right. \\
& \quad \times (\operatorname{sign}_0(z(s) - \hat{z}) + \operatorname{sign}_0(u(s) - \hat{u})) \chi_{\{x \in \Omega: z(s, x) = \hat{z}(x)\}} \psi \, dx \Big] ds \\
& + \int_{\hat{t}}^t \int_{\Omega} (f(s) - \hat{f}) H_n(u(s)) (\rho - \operatorname{sign}_0(z(s) - \hat{z})) \chi_{\{x \in \Omega: u(s, x) = \hat{u}(x)\}} \psi \, dx \, ds. \quad (55)
\end{aligned}$$

In order to prove (55), let us take in Lemma 4.2, $h(x) = \hat{u}(x) - k\rho(x)$, $\rho \in W^{1,p}(\Omega)$, $-1 \leq \rho \leq 1$, $k > 0$. Then, for any $\psi \in \mathcal{D}(\Omega)$, $0 \leq \psi \leq 1$,

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} \left(\int_{\hat{z}}^{z(t)} H_n((\gamma^{-1})^0(s)) \frac{1}{k} T_k((\gamma^{-1})^0(s) - \hat{u} + k\rho) \, ds \right) \psi \, dx \\
& + \int_{\Omega} (\mathbf{a}(x, Du(t)) - \mathbf{a}(x, D\hat{u})) \cdot D \left(H_n(u(t)) \frac{1}{k} T_k(u(t) - \hat{u} + k\rho) \right) \psi \, dx \\
& = \int_{\Omega} (f(t) - \hat{f}) H_n(u(t)) \frac{1}{k} T_k(u(t) - \hat{u} + k\rho) \psi \, dx \quad (56)
\end{aligned}$$

in $\mathcal{D}'([0, T])$. Integrating from \hat{t} to t , $0 < \hat{t} \leq t < T$,

$$\begin{aligned}
& \int_{\Omega} \int_{\hat{z}}^{z(t)} H_n((\gamma^{-1})^0(s)) \frac{1}{k} T_k((\gamma^{-1})^0(s) - \hat{u} + k\rho) \psi \, ds \, dx \\
& - \int_{\Omega} \int_{\hat{z}}^{z(\hat{t})} H_n((\gamma^{-1})^0(s)) \frac{1}{k} T_k((\gamma^{-1})^0(s) - \hat{u} + k\rho) \psi \, ds \, dx
\end{aligned}$$

$$\begin{aligned}
& + \int_{\hat{t}}^t \left[\int_{\Omega} (\mathbf{a}(x, Du(s)) - \mathbf{a}(x, D\hat{u})) D \left(H_n(u(s)) \frac{1}{k} T_k(u(s) - \hat{u} + k\rho) \psi \right) dx \right] ds \\
& = \int_{\hat{t}}^t \int_{\Omega} (f(s) - \hat{f}) H_n(u(s)) \frac{1}{k} T_k(u(s) - \hat{u} + k\rho) \psi dx ds.
\end{aligned} \tag{57}$$

For the first term in (57), we take limit in k and use that

$$\lim_{k \rightarrow 0} \frac{1}{k} T_k(r - q + k\rho) = \text{sign}_0(r - q) + \rho \chi_{\{r=q\}} \quad \forall -1 \leq \rho \leq 1, \tag{58}$$

and

$$\begin{aligned}
& \text{sign}_0(r - q) + \text{sign}_0(\hat{r} - \hat{q}) \chi_{\{r=q\}} \\
& = \text{sign}_0(\hat{r} - \hat{q}) + \text{sign}_0(r - q) \chi_{\{\hat{r}=\hat{q}\}} \quad \forall \hat{r} \in \gamma(r), \quad \hat{q} \in \gamma(q),
\end{aligned} \tag{59}$$

to obtain

$$\begin{aligned}
& \lim_{k \rightarrow 0} \int_{\Omega} \int_{\hat{z}}^{z(t)} H_n((\gamma^{-1})^0(s)) \frac{1}{k} T_k((\gamma^{-1})^0(s) - \hat{u} + k\rho) \psi ds dx \\
& = \int_{\Omega} \int_{\hat{z}}^{z(t)} H_n((\gamma^{-1})^0(s)) (\text{sign}_0((\gamma^{-1})^0(s) - \hat{u}) + \rho \chi_{\{s: (\gamma^{-1})^0(s)=\hat{u}\}}) \psi ds dx \\
& = \int_{\Omega} \left[\int_{\hat{z}}^{z(t)} H_n((\gamma^{-1})^0(s)) (\text{sign}_0(s - \hat{z}) + (\rho - \text{sign}_0(s - \hat{z})) \chi_{\{s: (\gamma^{-1})^0(s)=\hat{u}\}} \right. \\
& \quad \left. + \text{sign}_0((\gamma^{-1})^0(s) - \hat{u}) \chi_{\{s: s=\hat{z}\}}) \psi ds \right] dx \\
& = \int_{\Omega} \int_{\hat{z}}^{z(t)} H_n((\gamma^{-1})^0(s)) (\text{sign}_0(s - \hat{z}) + (\rho - \text{sign}_0(s - \hat{z})) \chi_{\{s: (\gamma^{-1})^0(s)=\hat{u}\}}) \psi ds dx \\
& = \int_{\Omega} \int_{\hat{z}}^{z(t)} H_n((\gamma^{-1})^0(s)) \text{sign}_0(s - \hat{z}) \psi ds dx \\
& \quad + \int_{\Omega} \int_{\hat{z}}^{z(t)} H_n((\gamma^{-1})^0(s)) (\rho - \text{sign}_0(s - \hat{z})) \chi_{\{s: (\gamma^{-1})^0(s)=\hat{u}\}} \psi ds dx.
\end{aligned}$$

Similarly, for the second term in (57),

$$\begin{aligned}
& \lim_{k \rightarrow 0} \left(- \int_{\Omega} \int_{\hat{z}}^{z(\hat{t})} H_n((\gamma^{-1})^0(s)) \frac{1}{k} T_k((\gamma^{-1})^0(s) - \hat{u} + k\rho) \psi \, ds \, dx \right) \\
&= - \int_{\Omega} \int_{\hat{z}}^{z(\hat{t})} H_n((\gamma^{-1})^0(s)) \operatorname{sign}_0(s - \hat{z}) \psi \, ds \, dx \\
&\quad - \int_{\Omega} \int_{\hat{z}}^{z(\hat{t})} H_n((\gamma^{-1})^0(s)) (\rho - \operatorname{sign}_0(s - \hat{z})) \chi_{\{s: (\gamma^{-1})^0(s) = \hat{u}\}} \psi \, ds \, dx.
\end{aligned}$$

Let us now decompose the third term in (57) as $D_1(k, n) + D_2(k, n) + D_3(k, n) + D_4(k, n)$, where

$$D_1(k, n) = \int_{\hat{t}}^t \int_{\Omega} (\mathbf{a}(x, Du(s)) - \mathbf{a}(x, D\hat{u})) \cdot Du(s) H'_n(u(s)) \frac{1}{k} T_k(u(s) - \hat{u} + k\rho) \psi \, dx \, ds,$$

$$D_2(k, n) = \int_{\hat{t}}^t \int_{\Omega} (\mathbf{a}(x, Du(s)) - \mathbf{a}(x, D\hat{u})) \cdot D\psi H_n(u(s)) \frac{1}{k} T_k(u(s) - \hat{u} + k\rho) \, dx \, ds,$$

$$\begin{aligned}
D_3(k, n) &= \int_{\hat{t}}^t \left[\int_{\Omega} (\mathbf{a}(x, Du(s)) - \mathbf{a}(x, D\hat{u})) \cdot (Du(s) - D\hat{u}) \right. \\
&\quad \left. \times H_n(u(s)) \frac{1}{k} T'_k(u(s) - \hat{u} + k\rho) \psi \, dx \right] ds
\end{aligned}$$

and

$$D_4(k, n) = \int_{\hat{t}}^t \int_{\Omega} (\mathbf{a}(x, Du(s)) - \mathbf{a}(x, D\hat{u})) \cdot D\rho H_n(u(s)) T'_k(u(s) - \hat{u} + k\rho) \psi \, dx \, ds.$$

Now, by the Dominated Convergence's Theorem, and using that $Du(s) = D\hat{u}$ when $u(s) = \hat{u}$,

$$\lim_{k \rightarrow 0} D_1(k, n) = \int_{\hat{t}}^t \int_{\Omega} (\mathbf{a}(x, Du(s)) - \mathbf{a}(x, D\hat{u})) \cdot Du(s) H'_n(u(s)) \operatorname{sign}_0(u(s) - \hat{u}) \psi \, dx \, ds,$$

$$\lim_{k \rightarrow 0} D_2(k, n) = \int_{\hat{t}}^t \int_{\Omega} (\mathbf{a}(x, Du(s)) - \mathbf{a}(x, D\hat{u})) \cdot D\psi H_n(u(s)) \operatorname{sign}_0(u(s) - \hat{u}) \, dx \, ds,$$

$$\lim_{k \rightarrow 0} D_4(k, n) = 0$$

and, using (H_3) , since $\hat{t} \leq t$,

$$D_3(k, n) \geq 0.$$

Finally, for the fourth term in (57), using (58) and (59), we have that

$$\begin{aligned} & \lim_{k \rightarrow 0} \int_{\hat{t}}^t \int_{\Omega} (f(s) - \hat{f}) H_n(u(s)) \frac{1}{k} T_k(u(s) - \hat{u} + k\rho) \psi \, dx \, ds \\ &= \int_{\hat{t}}^t \int_{\Omega} (f(s) - \hat{f}) H_n(u(s)) (\text{sign}_0(z(s) - \hat{z}) + \text{sign}_0(u(s) - \hat{u}) \chi_{\{x \in \Omega: z(s,x) = \hat{z}(x)\}}) \psi \, dx \, ds \\ & \quad + \int_{\hat{t}}^t \int_{\Omega} (f(s) - \hat{f}) H_n(u(s)) (\rho - \text{sign}_0(z(s) - \hat{z})) \chi_{\{x \in \Omega: u(s,x) = \hat{u}(x)\}} \psi \, dx \, ds. \end{aligned}$$

Hence, taking limit in (57) as k goes to 0, we get (55).

Step 2: Inequality up to $\partial\Omega$. For any $0 < \hat{t} \leq t < T$,

$$\begin{aligned} & \int_{\Omega} |z(t) - \hat{z}| \, dx - \int_{\Omega} |z(\hat{t}) - \hat{z}| \, dx + \int_{\partial\Omega} |w(t) - \hat{w}| \, d\sigma - \int_{\partial\Omega} |w(\hat{t}) - \hat{w}| \, d\sigma \\ & \leq \int_{\hat{t}}^t \left[\int_{\Omega} (f(s) - \hat{f}) (\text{sign}_0(z(s) - \hat{z}) + \text{sign}_0(u(s) - \hat{u}) \chi_{\{x \in \Omega: z(s,x) = \hat{z}(x)\}}) \, dx \right] ds \\ & \quad + \int_{\hat{t}}^t \left[\int_{\Omega} (f(s) - \hat{f}) (\text{sign}_0(z(t) - \hat{z}) - \text{sign}_0(z(s) - \hat{z})) \chi_{\{x \in \Omega: u(s,x) = \hat{u}(x)\}} \, dx \right] ds \\ & \quad + \int_{\hat{t}}^t \left[\int_{\partial\Omega} (g(s) - \hat{g}) (\text{sign}_0(w(s) - \hat{w}) + \text{sign}_0(u(s) - \hat{u}) \chi_{\{x \in \partial\Omega: w(s,x) = \hat{w}(x)\}}) \, d\sigma \right] ds \\ & \quad + \int_{\hat{t}}^t \left[\int_{\partial\Omega} (g(s) - \hat{g}) (\text{sign}_0(w(t) - \hat{w}) - \text{sign}_0(w(s) - \hat{w})) \chi_{\{x \in \partial\Omega: u(s,x) = \hat{u}(x)\}} \, d\sigma \right] ds. \end{aligned} \tag{60}$$

In fact, since in (55) there are no space derivatives of ρ , by approximation, we can take, for each t fixed, $\rho = \text{sign}_0(z(t) - \hat{z})$. Then, by monotonicity of sign_0 , the second term in (55) is positive and so, for any $0 < \hat{t} \leq t < T$,

$$\begin{aligned}
& \int_{\Omega} \int_{z(\hat{t})}^{z(t)} H_n((\gamma^{-1})^0(s)) \operatorname{sign}_0(s - \hat{z}) \psi \, ds \, dx \\
& + \int_{\hat{t}}^t \left[\int_{\Omega} (\mathbf{a}(x, Du(s)) - \mathbf{a}(x, D\hat{u})) \cdot Du(s) \right. \\
& \quad \left. \times H'_n(u(s)) \operatorname{sign}_0(u(s) - \hat{u}) \psi \, dx \right] ds + I \\
& \leq \int_{\hat{t}}^t \left[\int_{\Omega} (f(s) - \hat{f}) H_n(u(s)) (\operatorname{sign}_0(z(s) - \hat{z}) \right. \\
& \quad \left. + \operatorname{sign}_0(u(s) - \hat{u}) \chi_{\{x \in \Omega: z(s,x) = \hat{z}(x)\}}) \psi \, dx \right] ds \\
& \quad + \int_{\hat{t}}^t \left[\int_{\Omega} (f(s) - \hat{f}) H_n(u(s)) (\operatorname{sign}_0(z(t) - \hat{z}) \right. \\
& \quad \left. - \operatorname{sign}_0(z(s) - \hat{z})) \chi_{\{x \in \Omega: u(s,x) = \hat{u}(x)\}} \psi \, dx \right] ds, \tag{61}
\end{aligned}$$

where

$$\begin{aligned}
I &= \int_{\hat{t}}^t \int_{\Omega} (\mathbf{a}(x, Du(s)) - \mathbf{a}(x, D\hat{u})) \cdot D\psi H_n(u(s)) \operatorname{sign}_0(u(s) - \hat{u}) \, dx \, ds \\
&= \int_{\hat{t}}^t \int_{\Omega} (\mathbf{a}(x, Du(s)) - \mathbf{a}(x, D\hat{u})) \cdot D(\psi - 1) H_n(u(s)) \operatorname{sign}_0(u(s) - \hat{u}) \, dx \, ds.
\end{aligned}$$

Now, for $\hat{\rho} \in W^{1,p}(\Omega)$, $-1 \leq \hat{\rho} \leq 1$, proceeding as in Step 1 and using the fact that $\psi - 1 = -1$ on $\partial\Omega$, we obtain that

$$\begin{aligned}
I &\geq - \int_{\Omega} \int_{z(\hat{t})}^{z(t)} H_n((\gamma^{-1})^0(s)) \operatorname{sign}_0(s - \hat{z}) (\psi - 1) \, ds \, dx \\
&\quad - \int_{\Omega} \int_{z(\hat{t})}^{z(t)} H_n((\gamma^{-1})^0(s)) (\hat{\rho} - \operatorname{sign}_0(s - \hat{z})) \chi_{\{s: (\gamma^{-1})^0(s) = \hat{u}\}} (\psi - 1) \, ds \, dx \\
&\quad + \int_{\partial\Omega} \int_{w(\hat{t})}^{w(t)} H_n((\beta^{-1})^0(s)) \operatorname{sign}_0(s - \hat{w}) \, ds \, d\sigma
\end{aligned}$$

$$\begin{aligned}
& + \int_{\partial\Omega} \int_{w(\hat{t})}^{w(t)} H_n((\beta^{-1})^0(s)) (\hat{\rho} - \text{sign}_0(s - \hat{w})) \chi_{\{s: (\beta^{-1})^0(s) = \hat{u}\}} ds d\sigma \\
& + \int_{\hat{t}}^t \left[\int_{\Omega} (f(s) - \hat{f}) H_n(u(s)) \right. \\
& \quad \times (\text{sign}_0(z(s) - \hat{z}) + \text{sign}_0(u(s) - \hat{u})) \chi_{\{x \in \Omega: z(s,x) = \hat{z}(x)\}} (\psi - 1) dx \left. \right] ds \\
& + \int_{\hat{t}}^t \int_{\Omega} (f(s) - \hat{f}) H_n(u(s)) (\hat{\rho} - \text{sign}_0(z(s) - \hat{z})) \chi_{\{x \in \Omega: u(s,x) = \hat{u}(x)\}} (\psi - 1) dx ds \\
& - \int_{\hat{t}}^t \left[\int_{\partial\Omega} (g(s) - \hat{g}) H_n(u(s)) \right. \\
& \quad \times (\text{sign}_0(w(s) - \hat{w}) + \text{sign}_0(u(s) - \hat{u})) \chi_{\{x \in \partial\Omega: w(s,x) = \hat{w}(x)\}} d\sigma \left. \right] ds \\
& - \int_{\hat{t}}^t \int_{\partial\Omega} (g(s) - \hat{g}) H_n(u(s)) (\hat{\rho} - \text{sign}_0(w(s) - \hat{w})) \chi_{\{x \in \partial\Omega: u(s,x) = \hat{u}(x)\}} d\sigma ds \\
& - \int_{\hat{t}}^t \int_{\Omega} (\mathbf{a}(x, Du(s)) - \mathbf{a}(x, D\hat{u})) \cdot Du(s) H'_n(u(s)) \text{sign}_0(u(s) - \hat{u}) (\psi - 1) dx ds.
\end{aligned}$$

Therefore, from (61) we get

$$\begin{aligned}
& \int_{\Omega} \int_{z(\hat{t})}^{z(t)} H_n((\gamma^{-1})^0(s)) \text{sign}_0(s - \hat{z}) ds dx \\
& - \int_{\Omega} \int_{z(\hat{t})}^{z(t)} H_n((\gamma^{-1})^0(s)) (\hat{\rho} - \text{sign}_0(s - \hat{z})) \chi_{\{s: (\gamma^{-1})^0(s) = \hat{u}\}} (\psi - 1) ds dx \\
& + \int_{\partial\Omega} \int_{w(\hat{t})}^{w(t)} H_n((\beta^{-1})^0(s)) \text{sign}_0(s - \hat{w}) ds d\sigma \\
& + \int_{\partial\Omega} \int_{w(\hat{t})}^{w(t)} H_n((\beta^{-1})^0(s)) (\hat{\rho} - \text{sign}_0(s - \hat{w})) \chi_{\{s: (\beta^{-1})^0(s) = \hat{u}\}} ds d\sigma
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{\hat{t}}^t \int_{\Omega} (f(s) - \hat{f}) H_n(u(s)) (\text{sign}_0(z(s) - \hat{z}) + \text{sign}_0(u(s) - \hat{u}) \chi_{\{x \in \Omega: z(s,x) = \hat{z}(x)\}}) dx ds \\
&\quad + \int_{\hat{t}}^t \int_{\Omega} (f(s) - \hat{f}) H_n(u(s)) (\text{sign}_0(z(t) - \hat{z}) - \text{sign}_0(z(s) - \hat{z})) \chi_{\{x \in \Omega: u(s,x) = \hat{u}(x)\}} \psi dx ds \\
&\quad - \int_{\hat{t}}^t \int_{\Omega} (f(s) - \hat{f}) H_n(u(s)) (\hat{\rho} - \text{sign}_0(z(s) - \hat{z})) \chi_{\{x \in \Omega: u(s,x) = \hat{u}(x)\}} (\psi - 1) dx ds \\
&\quad + \int_{\hat{t}}^t \left[\int_{\partial\Omega} (g(s) - \hat{g}) H_n(u(s)) \right. \\
&\quad \times (\text{sign}_0(w(s) - \hat{w}) + \text{sign}_0(u(s) - \hat{u}) \chi_{\{x \in \partial\Omega: w(s,x) = \hat{w}(x)\}}) d\sigma \left. \right] ds \\
&\quad + \int_{\hat{t}}^t \int_{\partial\Omega} (g(s) - \hat{g}) H_n(u(s)) (\hat{\rho} - \text{sign}_0(w(s) - \hat{w})) \chi_{\{x \in \partial\Omega: u(s,x) = \hat{u}(x)\}} d\sigma ds \\
&\quad + \int_{\hat{t}}^t \int_{\Omega} (\mathbf{a}(x, Du(s)) - \mathbf{a}(x, D\hat{u})) \cdot Du(s) H'_n(u(s)) \text{sign}_0(u(s) - \hat{u}) dx ds.
\end{aligned}$$

Letting now n go to $+\infty$, on account of (6), we obtain

$$\begin{aligned}
&\int_{\Omega} |z(t) - \hat{z}| dx - \int_{\Omega} |z(\hat{t}) - \hat{z}| dx - \int_{\Omega} \int_{z(\hat{t})}^{z(t)} (\hat{\rho} - \text{sign}_0(s - \hat{z})) \chi_{\{s: (\gamma^{-1})^0(s) = \hat{u}\}} (\psi - 1) ds dx \\
&\quad + \int_{\partial\Omega} |w(t) - \hat{w}| d\sigma - \int_{\partial\Omega} |w(\hat{t}) - \hat{w}| d\sigma \\
&\quad + \int_{\partial\Omega} \int_{w(\hat{t})}^{w(t)} (\hat{\rho} - \text{sign}_0(s - \hat{w})) \chi_{\{s: (\beta^{-1})^0(s) = \hat{u}\}} ds d\sigma \\
&\leq \int_{\hat{t}}^t \left[\int_{\Omega} (f(s) - \hat{f}) (\text{sign}_0(z(s) - \hat{z}) + \text{sign}_0(u(s) - \hat{u}) \chi_{\{x \in \Omega: z(s,x) = \hat{z}(x)\}}) dx \right] ds \\
&\quad + \int_{\hat{t}}^t \left[\int_{\Omega} (f(s) - \hat{f}) (\text{sign}_0(z(t) - \hat{z}) - \text{sign}_0(z(s) - \hat{z})) \chi_{\{x \in \Omega: u(s,x) = \hat{u}(x)\}} \psi dx \right] ds
\end{aligned}$$

$$\begin{aligned}
& - \int_{\hat{t}}^t \int_{\Omega} (f(s) - \hat{f})(\hat{\rho} - \text{sign}_0(z(s) - \hat{z})) \chi_{\{x \in \Omega: u(s,x) = \hat{u}(x)\}} (\psi - 1) dx ds \\
& + \int_{\hat{t}}^t \left[\int_{\partial\Omega} (g(s) - \hat{g})(\text{sign}_0(w(s) - \hat{w}) + \text{sign}_0(u(s) - \hat{u})) \chi_{\{x \in \partial\Omega: w(s,x) = \hat{w}(x)\}} d\sigma \right] ds \\
& + \int_{\hat{t}}^t \int_{\partial\Omega} (g(s) - \hat{g})(\hat{\rho} - \text{sign}_0(w(s) - \hat{w})) \chi_{\{x \in \partial\Omega: u(s,x) = \hat{u}(x)\}} d\sigma ds. \tag{62}
\end{aligned}$$

Taking into (62) ψ_m instead of ψ such that $L^1(\Omega)$ - $\lim_m \psi_m = 1$ and letting m go to $+\infty$, we have

$$\begin{aligned}
& \int_{\Omega} |z(t) - \hat{z}| dx - \int_{\Omega} |z(\hat{t}) - \hat{z}| dx + \int_{\partial\Omega} |w(t) - \hat{w}| d\sigma - \int_{\partial\Omega} |w(\hat{t}) - \hat{w}| d\sigma \\
& + \int_{\partial\Omega} \int_{w(\hat{t})}^{w(t)} (\hat{\rho} - \text{sign}_0(s - \hat{w})) \chi_{\{s: (\beta^{-1})^0(s) = \hat{u}\}} ds d\sigma \\
& \leq \int_{\hat{t}}^t \left[\int_{\Omega} (f(s) - \hat{f})(\text{sign}_0(z(s) - \hat{z}) + \text{sign}_0(u(s) - \hat{u})) \chi_{\{x \in \Omega: z(s,x) = \hat{z}(x)\}} dx \right] ds \\
& + \int_{\hat{t}}^t \left[\int_{\Omega} (f(s) - \hat{f})(\text{sign}_0(z(t) - \hat{z}) - \text{sign}_0(z(s) - \hat{z})) \chi_{\{x \in \Omega: u(s,x) = \hat{u}(x)\}} dx \right] ds \\
& + \int_{\hat{t}}^t \left[\int_{\partial\Omega} (g(s) - \hat{g})(\text{sign}_0(w(s) - \hat{w}) + \text{sign}_0(u(s) - \hat{u})) \chi_{\{x \in \partial\Omega: w(s,x) = \hat{w}(x)\}} dx \right] ds \\
& + \int_{\hat{t}}^t \int_{\partial\Omega} (g(s) - \hat{g})(\hat{\rho} - \text{sign}_0(w(s) - \hat{w})) \chi_{\{x \in \partial\Omega: u(s,x) = \hat{u}(x)\}} d\sigma ds. \tag{63}
\end{aligned}$$

Now, by approximation, we can take, for each t fixed, $\hat{\rho}$ such that its trace is equal to $\text{sign}_0(w(t) - \hat{w})$. Then the fifth term in the above expression is positive and (60) follows.

Step 3: Integral solution. Let

$$\varphi_1(t) := \int_{\Omega} |z(t) - \hat{z}| dx + \int_{\partial\Omega} |w(t) - \hat{w}| d\sigma,$$

$$\begin{aligned}\varphi_2(s) &:= \int_{\Omega} (f(s) - \hat{f})(\text{sign}_0(z(s) - \hat{z}) + \text{sign}_0(u(s) - \hat{u})) \chi_{\{x \in \Omega: z(s,x) = \hat{z}(x)\}} dx \\ &\quad + \int_{\partial\Omega} (g(s) - \hat{g})(\text{sign}_0(w(s) - \hat{w}) + \text{sign}_0(u(s) - \hat{u})) \chi_{\{x \in \partial\Omega: w(s,x) = \hat{w}(x)\}} d\sigma\end{aligned}$$

and

$$\begin{aligned}\varphi_3(t, s) &:= \int_{\Omega} (f(s) - \hat{f})(\text{sign}_0(z(t) - \hat{z}) - \text{sign}_0(z(s) - \hat{z})) \chi_{\{x \in \Omega: u(s,x) = \hat{u}(x)\}} dx \\ &\quad + \int_{\partial\Omega} (g(s) - \hat{g})(\text{sign}_0(w(t) - \hat{w}) - \text{sign}_0(w(s) - \hat{w})) \chi_{\{x \in \partial\Omega: u(s,x) = \hat{u}(x)\}} d\sigma.\end{aligned}$$

Taking in (60) $\hat{t} = t - h$, $h > 0$, dividing by h and letting h go to 0, we get for any $\eta \in \mathcal{D}([0, T])$, $\eta \geq 0$,

$$\begin{aligned}- \int_0^T \varphi_1(t) \eta_t(t) dt &= - \lim_{h \rightarrow 0^+} \int_0^T \varphi_1(t) \frac{\eta(t+h) - \eta(t)}{h} dt \\ &= \lim_{h \rightarrow 0^+} \int_0^T \frac{\varphi_1(t) - \varphi_1(t-h)}{h} \eta(t) dt \\ &\leq \lim_{h \rightarrow 0^+} \left(\int_0^T \frac{1}{h} \left(\int_{t-h}^t \varphi_2(s) ds \right) \eta(t) dt + \int_0^T \frac{1}{h} \left(\int_{t-h}^t \varphi_3(t, s) ds \right) \eta(t) dt \right).\end{aligned}\tag{64}$$

By the Dominated Convergence Theorem,

$$\begin{aligned}\lim_{h \rightarrow 0^+} \int_0^T \frac{1}{h} \left(\int_{t-h}^t \varphi_2(s) ds \right) \eta(t) dt &= - \lim_{h \rightarrow 0^+} \int_0^T \left(\int_0^t \varphi_2(s) ds \right) \frac{\eta(t+h) - \eta(t)}{h} dt \\ &= - \int_0^T \left(\int_0^t \varphi_2(s) ds \right) \eta_t(t) dt = \int_0^T \varphi_2(t) \eta(t) dt.\end{aligned}$$

On the other hand, for h small enough,

$$\int_0^T \frac{1}{h} \left(\int_{t-h}^t \varphi_3(t, s) ds \right) \eta(t) dt = \int_0^T \frac{1}{h} \left(\int_s^{s+h} \varphi_3(t, s) \eta(t) dt \right) ds$$

and

$$\begin{aligned}
& \left| \int_0^T \frac{1}{h} \left(\int_s^{s+h} \varphi_3(t, s) \eta(t) dt \right) ds \right| \\
& \leq \int_0^T \frac{1}{h} \left(\int_s^{s+h} \int_{\Omega} |f(s) - \hat{f}| |\text{sign}_0(z(t) - \hat{z}) - \text{sign}_0(z(s) - \hat{z})| |\eta(t)| dx dt \right) ds \\
& \quad + \int_0^T \frac{1}{h} \left(\int_s^{s+h} \int_{\partial\Omega} |g(s) - \hat{g}| |\text{sign}_0(w(t) - \hat{w}) - \text{sign}_0(w(s) - \hat{w})| |\eta(t)| d\sigma dt \right) ds \\
& \leq \|\eta\|_{L^\infty(0, T)} \int_0^T \int_{\Omega} |f(s) - \hat{f}| \frac{1}{h} \int_s^{s+h} |\text{sign}_0(z(t) - \hat{z}) - \text{sign}_0(z(s) - \hat{z})| dt dx ds \\
& \quad + \|\eta\|_{L^\infty(0, T)} \int_0^T \left[\int_{\partial\Omega} |g(s) - \hat{g}| \frac{1}{h} \int_s^{s+h} |\text{sign}_0(w(t) - \hat{w}) - \text{sign}_0(w(s) - \hat{w})| dt d\sigma \right] ds.
\end{aligned}$$

Now, since $(t, x) \mapsto \text{sign}_0(z(t, x) - \hat{z}(x)) \in L^1(Q_T)$, if we set

$$\varrho_h(s, x) = \frac{1}{h} \int_s^{s+h} |\text{sign}_0(z(t, x) - \hat{z}(x)) - \text{sign}_0(z(s, x) - \hat{z}(x))| dt,$$

we have that

$$\lim_{h \rightarrow 0^+} \varrho_h(s, \cdot) = 0 \quad \text{in } L^1(\Omega) \text{ a.e. } s \in [0, T].$$

Moreover,

$$\varrho_h(s, x) \leq 2 \quad \text{a.e. in } Q_T.$$

Consequently, applying twice the Dominated Convergence Theorem, we get

$$\lim_{h \rightarrow 0^+} \int_0^T \int_{\Omega} |f(s) - \hat{f}| \frac{1}{h} \int_s^{s+h} |\text{sign}_0(z(t) - \hat{z}) - \text{sign}_0(z(s) - \hat{z})| dt dx ds = 0.$$

Similarly,

$$\lim_{h \rightarrow 0^+} \int_0^T \int_{\partial\Omega} |g(s) - \hat{g}| \frac{1}{h} \int_s^{s+h} |\text{sign}_0(w(t) - \hat{w}) - \text{sign}_0(w(s) - \hat{w})| dt d\sigma ds = 0.$$

Therefore, from (64) we obtain that

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} |z(t) - \hat{z}| dx + \frac{d}{dt} \int_{\partial\Omega} |w(t) - \hat{w}| d\sigma \\
& \leq \int_{\Omega} (f(t) - \hat{f}) \operatorname{sign}_0(z(t) - \hat{z}) dx + \int_{\{x \in \Omega: z(t) = \hat{z}\}} |f(t) - \hat{f}| dx \\
& \quad + \int_{\partial\Omega} (g(t) - \hat{g}) \operatorname{sign}_0(w(t) - \hat{w}) d\sigma + \int_{\{x \in \partial\Omega: w(t) = \hat{w}\}} |g(t) - \hat{g}| d\sigma
\end{aligned}$$

in $\mathcal{D}'(]0, T[)$, and the proof of Theorem 4.3 is finished. \square

Acknowledgments

We would like to thank the referee for many interesting remarks about the paper that have improved its readability and simplified some arguments in the proofs. This work has been performed during the visits of the first, third and fourth authors to the Université de Picardie Jules Verne and the visit of the second author to the Universitat de València. They thank these institutions for their support and hospitality. The first, third and fourth authors have been partially supported by the Spanish MEC and FEDER, project MTM2005-00620, and by the Generalitat Valenciana, project ACOMP06/196.

References

- [1] T. Aiki, Multi-dimensional two-phase Stefan problems with nonlinear dynamic boundary conditions, in: *Nonlinear Analysis and Applications*, Warsaw, 1994, in: GAKUTO Internat. Ser. Math. Sci. Appl., vol. 7, Gakkōtoshō, Tokyo, 1996, pp. 1–25.
- [2] H.W. Alt, S. Luckhaus, Quasilinear elliptic–parabolic differential equations, *Math. Z.* 183 (1983) 311–341.
- [3] K. Ammar, P. Wittbold, Existence of renormalized solutions of degenerate elliptic–parabolic problems, *Proc. Roy. Soc. Edinburgh Sect. A* 133 (2003) 477–496.
- [4] F. Andreu, N. Igbida, J.M. Mazón, J. Toledo, L^1 existence and uniqueness results for quasi-linear elliptic equations with nonlinear boundary conditions, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 24 (2007) 61–89.
- [5] F. Andreu, N. Igbida, J.M. Mazón, J. Toledo, A degenerate elliptic–parabolic problem with nonlinear dynamical boundary conditions, *Interfaces Free Bound.* 8 (2006) 447–479.
- [6] F. Andreu, J.M. Mazón, S. Segura de León, J. Toledo, Quasi-linear elliptic and parabolic equations in L^1 with nonlinear boundary conditions, *Adv. Math. Sci. Appl.* 7 (1997) 183–213.
- [7] F. Andreu, J.M. Mazón, S. Segura, J. Toledo, Existence and uniqueness for a degenerate parabolic equation with L^1 -data, *Trans. Amer. Math. Soc.* 315 (1999) 285–306.
- [8] B. Andreianov, N. Igbida, Revising uniqueness for a nonlinear diffusion–convection equation, *J. Differential Equations* 227 (2006) 69–79.
- [9] G. Astarita, G. Marrucci, *Principles of Non-Newtonian Fluid Mechanics*, McGraw–Hill, New York, 1974.
- [10] A. Bamberger, Étude d’une équation doublement nonlinéaire, *J. Funct. Anal.* 24 (1977) 148–155.
- [11] I. Bejenaru, J.I. Diaz, I. Vrabie, An abstract approximate controllability result and applications to elliptic and parabolic system with dynamics boundary conditions, *Electron. J. Differential Equations* 50 (2001) 1–19.
- [12] Ph. Bénilan, *Equations d’évolution dans un espace de Banach quelconque et applications*, thesis, Univ. Orsay, 1972.
- [13] Ph. Bénilan, L. Boccardo, Th. Gallouët, R. Gariepy, M. Pierre, J.L. Vázquez, An L^1 -theory of existence and uniqueness of solutions of nonlinear elliptic equations, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)* 22 (1995) 241–273.
- [14] Ph. Bénilan, M.G. Crandall, A. Pazy, *Evolution Equations Governed by Accretive Operators*, in press.
- [15] Ph. Bénilan, P. Wittbold, On mild and weak solutions of elliptic–parabolic problems, *Adv. Differential Equations* 1 (1996) 1053–1073.
- [16] D. Blanchard, F. Murat, Renormalized solutions of nonlinear parabolic problems with L^1 data, *Proc. Roy. Soc. Edinburgh* 127 (1997) 1137–1152.

- [17] D. Blanchard, A. Porretta, Stefan problems with nonlinear diffusion and convection, *J. Differential Equations* 210 (2005) 383–428.
- [18] L. Boccardo, F. Gallouët, J.P. Puel, Existence of bounded solutions for nonlinear elliptic unilateral problems, *Ann. Mat. Pura Appl.* 152 (1988) 183–196.
- [19] H. Brezis, Problèmes unilatéraux, *J. Math. Pures Appl.* 51 (1972) 1–168.
- [20] H. Brezis, *Opérateur Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*, Oxford Univ. Press, Oxford, 1984.
- [21] J. Carrillo, Entropy solutions for nonlinear degenerate problems, *Arch. Ration. Mech. Anal.* 147 (1999) 269–361.
- [22] J. Carrillo, P. Wittbold, Uniqueness of renormalized solutions of degenerate elliptic–parabolic problems, *J. Differential Equations* 156 (1999) 93–121.
- [23] P. Colli, J.F. Rodrigues, Diffusion through thin layers with high specific heat, *Asymptot. Anal.* 3 (1990) 249–263.
- [24] M.G. Crandall, An introduction to evolution governed by accretive operators, in: L. Cesari, et al. (Eds.), *Dynamical Systems, An International Symposium*, vol. 1, Academic Press, New York, 1976, pp. 131–165, Dekker, New York, 1991.
- [25] J. Crank, *Free and Moving Boundary Problems*, North-Holland, Amsterdam, 1977.
- [26] E. DiBenedetto, A. Friedman, The ill-posed Hele–Shaw model and the Stefan problem for supercooler water, *Trans. Amer. Math. Soc.* 282 (1984) 183–204.
- [27] R. Di Perna, P.L. Lions, On the Cauchy problem for Boltzmann equations: Global existence and weak stability, *Ann. of Math.* 130 (1989) 321–366.
- [28] G. Duvaux, J.L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, 1976.
- [29] C.M. Elliot, V. Janosky, A variational inequality approach to the Hele–Shaw flow with a moving boundary, *Proc. Roy. Soc. Edinburgh Sect. A* 88 (1981) 93–107.
- [30] J. Escher, Quasilinear parabolic systems with dynamical boundary conditions, *Comm. Partial Differential Equations* 18 (1993) 1309–1364.
- [31] N. Igbida, The mesa-limit of the porous medium equation and the Hele–Shaw problem, *Differential Integral Equations* 15 (2002) 129–146.
- [32] N. Igbida, A nonlinear diffusion problem with localized large diffusion, *Comm. Partial Differential Equations* 29 (5–6) (2004) 647–670.
- [33] N. Igbida, M. Kirane, A degenerate diffusion problem with dynamical boundary conditions, *Math. Ann.* 323 (2002) 377–396.
- [34] N. Igbida, J.M. Urbano, Uniqueness for nonlinear degenerate problems, *NoDEA Nonlinear Differential Equations Appl.* 10 (2003) 287–307.
- [35] N. Igbida, P. Wittbold, Renormalized solutions for Stefan type problems: Existence and uniqueness, preprint.
- [36] N. Kenmochi, Neumann problems for a class of nonlinear degenerate parabolic equations, *Differential Integral Equations* 2 (1990) 253–273.
- [37] L. Kondic, P. Palffy-Muhoray, M.J. Shelley, Models of non-Newtonian Hele–Shaw flow, *Phys. Rev. E* 54 (1996) 4536–4539.
- [38] L. Kondic, M.J. Shelley, P. Palffy-Muhoray, Non-Newtonian Hele–Shaw flow and the Saffman–Taylor instability, *Phys. Rev. Lett.* 80 (1998) 1433–1437.
- [39] R. Landes, On the existence of weak solutions for quasilinear parabolic initial boundary value problems, *Proc. Roy. Soc. Edinburgh Sect. A* 89 (1981) 217–237.
- [40] G.M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal. TMA* 12 (1988) 1203–1219.
- [41] J.L. Lions, *Quelques méthodes de résolution de problèmes aux limites nonlinéaires*, Dunod–Gauthier–Villars, Paris, 1968.
- [42] M. Primicerio, J.F. Rodrigues, The Hele–Shaw problem with nonlocal injection condition, in: *Nonlinear Mathematical Problems in Industry*, II, Iwaki, 1992, in: *GAKUTO Internat. Ser. Math. Sci. Appl.*, vol. 2, Gakkōtoshō, Tokyo, 1993, pp. 375–390.
- [43] J.F. Rodrigues, V.A. Solonnikov, F. Yi, On a parabolic system with time derivative in the boundary conditions and related free boundary problems, *Math. Ann.* 315 (1999) 61–91.
- [44] N. Su, Multidimensional degenerate diffusion problem with evolutionary boundary condition: Existence, uniqueness, and approximation, in: *Flow in Porous Media*, Oberwolfach, 1992, in: *Internat. Ser. Numer. Math.*, vol. 114, Birkhäuser, Basel, 1993, pp. 165–178.
- [45] I.I. Vrabie, *Compactness Methods for Nonlinear Evolutions*, second ed., Pitman Monogr. Surveys Pure Appl. Math., vol. 75, Longman Scientific & Technical, Harlow, 1995.